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Deterministic feedback stabilization of uncertain dynamical systems

Goodall, David Peter

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DETERMINISTIC FEEDBACK STABILIZATION OF UNCERTAIN DYNAMICAL SYSTEMS

submitted by David Peter Goodall

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D. Goodall.

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SUMMARY

Deterministic design of stabilizing feedback controls for classes of imperfectly known dynamical systems is considered via an approach which does not require full identification of the uncertain elements. Initially, systems modelled by differential equations are investigated and, based only on knowledge of functional properties and bounds relating to the uncertainty, a class of nonlinear feedback controls is developed which guarantees global uniform asymptotic stability of a compact set, containing the state origin. The class of controls is then extended for problems of tracking and model-following in the presence of uncertainty.

A problem formulation based on differential inclusions is subsequently adopted. Here, system uncertainty is modelled by set-valued maps. A class of generalized feedback controls is described, defined in terms of set-valued maps with practical analogues in the form of discontinuous feedbacks, which guarantee global uniform asymptotic stability of a compact set, containing the state origin, for the differential inclusion model. Moreover, under a matched uncertainty hypothesis, the class of generalized controls guarantees global uniform asymptotic stability of the zero state and ultimate attainment of prescribed model behaviour. This problem formulation is then extended to include problems of tracking and model-following.

Finally, a class of generalized feedback controls is presented which guarantees global asymptotic stability of the zero state of a class of nonlinearly coupled uncertain dynamical systems; the uncertainty in the system being modelled by set-valued maps. The uncertain dynamical system is based on a prototype sytem that has the structure of two bilinearly coupled subsystems and has a non-asymptotically-stabilizable linearization.

1.GENERAL DESCRIPTION OF THE PROBLEM

1.1 Introduction

In this first chapter, uncertain dynamical systems are described and a methodology of feedback control to stabilize uncertain dynamical systems is introduced. Initially, the uncertain systems are modelled by differential equations but, later, these models are generalized in the sense that the equations are replaced by inclusions. A class of nonlinear uncertain systems, to be stabilized by feedback, is described in §1.2. In addition, an overview of some techniques that are used for the control of uncertain dynamical systems is given. The system model, described in §1.2, is then generalized in §1.3 to take account of the fact that the true controlled vector field may be imprecisely known and, also, the controlled vector field may be discontinuous. In §1.4 the stabilization problem is stated and appropriate techniques for feedback stabilization are described. Finally in §1.5, notation and mathematical preliminaries are introduced.

1.2 Control of uncertain dynamical systems

In this thesis only continuous-time dynamical systems will be considered. In particular, attention will be resticted to control systems in which the governing equations that model the dynamics of the system are based on ordinary differential equations or differential inclusions. For example, a dynamical system of the form

$$\dot{x}(t) = H(t,x(t),u(t)) , \qquad x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m \qquad (1.2.1)$$

where $x(t)$ is the state vector (representing dynamic quantities in the system), u(t) is the control or input vector, $1 \leq m \leq n$, and t denotes time, is to be considered. Note that non-autonomous systems, i.e. systems for which the function H depends explicitly on time t, will be investigated.

Suppose (1.2.1) corresponds to a mathematical model of a "real world" process then, almost surely, some approximation, imprecision or uncertainty will have been introduced during the modelling procedure; the function H is, at best, a "reasonable" representation of the true controlled vector field. If (1.2.1) is deemed to be a reasonable model then the theories of classical and controlled differential equations may provide the appropriate framework for analysis and control design. Note, however, that if control synthesis is an objective, then discontinuous feedback ($u(t)= D(t,x(t))$ is a natural candidate in many problems of stabilization and optimization, in which case the associated differential equation ($\dot{x}(t) = H_D(t,x(t))$; $H_D(t,x) := H(t,x,D(t,x))$), modelling the feedback controlled system, fails to satisfy the requisite hypotheses of the classical theory. Returning to the basic modelling problem, in many cases the determination of an acceptably accurate model of the form (1.2.1) is impossible, i.e. uncertainty may be an intrinsic feature. For example, it may only be possible to determine a model structure which is specified up to a collection of parameters, the values of which are unknown (but possibly restricted to known sets); furthermore, realistic processes are frequently subject to extraneous disturbances (again possibly with known bounds) and, in addition, there may be imperfectly known inputs. Under such imperfect knowledge of the mathematical model, one seeks to design a (feedback) controller such that the system exhibits some desired behaviour.

To obtain desired system response, two approaches have been widely used. Firstly, it may be possible to regard the problem from a stochastic point of view (see, for example, Åström [6], Willems and Willems [85]). In this case, the "randomness" in the model is assumed to have a statistical characterization and the desired behaviour of the system is descibed in a statistical sense, for which stochastic control theory is appropriate. On the other hand, if structural properties and bounds relating to the uncertainty are known, then a deterministic treatment may be feasible. In this case, one desires some guaranteed performance of the dynamical system. Here, in this thesis, the deterministic approach is adopted.

Deterministic control of uncertain dynamical systems has been the focus of much research. Most of the research has been conducted in two main areas. In the first, the bounds on the uncertainties of a prescribed system are studied for which desired behaviour is preserved for a given controller, i.e. assumptions are made on the allowable sizes of the uncertainties. This is often characterized as a "robustness" problem for a system . In particular, the "robust stability" problem concerns the extent to which a nominal system remains stable when subject to a certain class of perturbations (see Chen and Desoer [16], Hinrichsen and Pritchard [37], [38], MacFarlane and Postlethwaite [55], Pritchard and Townley [60], Safonov [69], Safonov and Athens [72], Vidyasagar, Schneider and Francis [81], Zames [91]). More recently this work has developed into H^2 and H^∞ design problems (see Glover [27], Vidyasagar [80], and Youla and Bongiorno [88]). Here, the H^2 and H^∞ norms are very useful in characterizing robustness with respect to the frequency domain, through the optimization of some performance measure. Other norms can be used, for example, Safonov [70] used L^∞ optimal control theory to analyze the robustness of a control system. The basic design problem is to synthesize controllers that minimize the appropriate norm of the transfer function of a feedback system and, at the same time, stabilize the given nominal feedback system subject to uncertain perturbations with known bounds. In general, these techniques are used to treat unstructured system uncertainties; however, these techniques can be adapted for systems with some type of structural uncertainty (see Safonov [71]).

In the second main area, the approach is to synthesize a controller, under assumptions (structural in nature) about the uncertainties, in order to assure the desired behaviour (see, for example, Barmish, Corless and Leitmann [9], Barmish and Leitmann [10], Chen [17], Corless, Goodall, Leitmann, and Ryan [20], Corless and Leitmann [21], Gutman [31], Gutman and Palmor [33], Leitmann [49]–[51], Ryan [63], Ryan and Corless [67]). The assumptions concerning the uncertainties in a system are structural in nature in the sense that certain prescribed conditions relating to the uncertainty in the system, usually referred to as "matching conditions", must be satisfied. Much of this work concerns "stability" properties and the analysis is performed in the time domain.

Other techniques for the design of controls to stabilize uncertain systems are: (*i*) *Variable structure systems* (for a description see Itkis [40] and Utkin [78], [79]) which is based on the concept of an "attractive" subspace, on which it is possible to guarantee certain dynamic behaviour. This theory has been applied, in particular, to model-following and model reference adaptive control (see, for example, Zinober [92] and Zinober, El-Ghezawi and Billings [93]). (*ii*) *Quantitative feedback theory* (see Horowitz [39] and Yaniv and Horowitz [86]) which uses quantitative design methods to guarantee that desired performance tolerances are achieved over a given uncertainty range, whilst stabilizing the perturbed system. (*iii*) *Hurwitz-condition* approach, which is based on the well known Hurwitz stability criterion for linear systems, deals with structured parameter variations (see, in particular, Keel and Bhattacharyya [43] and Wei and Barmish [83]). (*iv*) *Game-theoretic* (or *minimax*) approach which views uncertain parameters as antagonists that maximize a performance measure, being minimized by the control function. For an application to a discrete-time uncertain dynamical system, see Bertsekas and Rhodes [11], who use dynamic programming concepts.

1.3 A differential inclusion model for an uncertain system

As mentioned in the previous section, the uncertain system (1.2.1) is characterized by the function H which, due to the uncertainties in the system, may be unknown. In particular, for each t, it is assumed that $H(t,x(t),u(t))$ is unknown, but that the set of all possible $H(t,x(t),u(t))$ is known and compact. Thus, in order to handle the system uncertainty, it is more natural to adopt a model where the function H is replaced by a known ***set-valued map*** or ***multifunction***, $\mathcal{H}$. The notation $\mathcal{H}$: $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is used to denote that $\mathcal{H}$ maps $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$ to the subsets of $\mathbb{R}^n$. Therefore, the controlled uncertain system (1.2.1) is more generally defined via a controlled differential inclusion

$$\dot{x}(t) \in \mathcal{H}(t,x(t),u(t)), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, \tag{1.3.1}$$

where $(t,x,u) \mapsto \mathcal{H}(t,x,u)$, $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \rightrightarrows \mathbb{R}^n$, is a known multifunction, on which additional structure will be imposed later. In essence, $\mathcal{H}(t,x(t),u(t))$ is the set of all possible "velocities" $\dot{x}(t)$ of the uncertain system at time t.

Problems of feedback control naturally lead to discontinuous vector fields. However, as noted in the paper by Monopoli [58], discontinuous feedback controls generate technical difficulties on the existence of solutions to differential equations. Instead of replacing the control with a continuous approximation, an alternative approach is to model the discontinuous control function by a set-valued function. With this approach, the class of controls for the system consist of ***generalized*** feedbacks (see definition 3.4.1 for a precise definition). These feedbacks, which relate the state of the system to the controls, are set-valued maps associating a set of control values with each state of the system. Given a generalized feedback $\mathcal{F}(t,x(t))$, the differential inclusion system :

$$\dot{x}(t) \in \mathcal{H}_{\mathcal{F}}(t,x(t)), \tag{1.3.2}$$

where

$$\mathcal{H}_{\mathcal{F}}(t,x) := \mathcal{H}(t,x,\mathcal{F}(t,x)) = \bigcup_{u \in \mathcal{F}(t,x)} \mathcal{H}(t,x,u)$$

will be investigated. Stabilization properties for controlled differential inclusion systems have been investigated by Goodall and Ryan [28]–[29], Gutman [31], Gutman and Palmor [33], and Leitmann [49].

An important requirement is that (1.3.2) must have existence of solutions. In order to fulfill this requirement, it is assumed that H_F has convex, as well as compact, values. Also, some concept of continuity must be ascribed to H_F. In this thesis, it is assumed that H_F is ***upper semicontinuous*** (defined in §1.5). These hypotheses can be varied. The convexity assumption was relaxed by Filippov [25], who assumed that H_F was continuous. Kaczyński and Olech [41] and Antosiewicz and Cellina [5] extended the work of Filippov to the case where H_F satisfied some "Carathéodory" type conditions (see §1.5). Łojasiewicz [52] replaced the assumptions of upper semicontinuity and convex values with the assumption of lower semicontinuity (for a definition of lower semicontinuity see Aubin and Cellina [7]). Recently, Himmelberg and Van Vleck [36] have extended the above results to the case when H_F is non-compact.

1.4 Techniques for stabilization

Initially, the stability of uncertain systems of the form (1.2.1), will be investigated (see chapter 2). In chapter 3, a framework is then developed so that the stability of more general uncertain systems, modelled by controlled differential inclusions of the form (1.3.1), can be analyzed. More precise definitions of various types of stability are given in chapter 2.

In this thesis, stabilization by feedback is considered. When synthesizing feedback controls for stabilization problems, it naturally leads to controls that are discontinuous. However, to overcome any difficulties obtained through practical implementation of controls of this type, continuous approximations to the discontinuous controls can be made. Hence, feedback stabilization will be investigated using both continuous and discontinuous controls. In chapter 2, continuous controls are used to stabilize uncertain dynamical systems of the form

(1.2.1), whilst in chapter 3 continuous feedback controls are used to stabilize differential inclusion systems of the form (1.3.1). A class of discontinuous feedback controls, modelled by set-valued functions, is introduced in chapter 4 and these set-valued functions are used to stabilize uncertain differential inclusion systems described by (1.3.2). Finally, in chapter 5, discontinuous feedback controls are used to stabilize a class of nonlinearly coupled uncertain dynamical systems.

Two deterministic techniques will be used for the stability analysis. The first technique for analyzing stability of a nonlinear system (either controlled or uncontrolled) is the *second* (or *direct*) method of A.M. Lyapunov, which was published in 1892, translated into French in 1907 [53] and reprinted in English in 1949 [54]. The method is general in the sense that it is not restricted to a particular class of differential equations. Also, solutions of the differential equations are not required, i.e. it is a qualitative technique. Lyapunov's method is a very powerful method used to address stability problems for both linear and nonlinear, nonautonomous systems since the technique can be used to examine boundedness and asymptotic behaviour of solutions as well as stability. An elementary introduction to Lyapunov's second method is given by LaSalle and Lefschetz [47]. The basic idea, for the synthesis problem, is to choose an appropriate positive definite function, V say, and then construct a control function such that V decreases along all trajectories of the system. This involves investigating the time derivative, $\dot{V}$, along all trajectories. In some work by Kalman and Bertram [42], it is shown that this can lead to discontinuous control functions.

The second technique, called *Variable Structure Systems* theory, originated from an idea first introduced by Fillipov [24] and has been used to stabilize a class of nonlinear systems. For a description of this technique see Itkis [40] and Utkin [78], [79]. This technique is based on the concept of an "attractive" subspace, on which it is possible to guarantee certain desired dynamic behaviour. This subspace is "attractive" in the sense that neighbouring system trajectories are drawn onto the subspace and, subsequently, the system motion is constrained to remain on (or "slide along") the subspace by exploiting the invariance properties of "sliding modes". Thus, whilst in "sliding mode", a trajectory is subject to constraints on the dynamics relating to the subspace. Furthermore, if the equations governing the dynamics are perturbed, the constraints on the dynamics remain the same so long as the subspace is attractive for the perturbed equation. Since attractivity is a local property, it is necessary to introduce extra hypotheses to ensure that all system trajectories eventually attain the attractive subspace, i.e. to ensure that the subspace is globally attractive. The importance of this theory for uncertain dynamical systems is that the system motion on the subspace is unaffected by bounded uncertainties and disturbances. This theory also leads to discontinuous control action. However, discontinuous control action imposes practical difficulties with respect to implementation of the control and, also, analytical difficulties with respect to the existence of solutions to the differential equations modelling the system. Thus, as mentioned earlier, the practical difficulties can be overcome by using continuous approximations to the discontinuous controls. For example, a relay could be replaced by a saturated linear control which is continuous. To overcome the analytical difficulties, the discontinuous control function can be modelled as a multifunction and the theory relating to differential inclusions can be invoked (see chapters 3 and 4).

Thus, with respect to stability of uncertain dynamical systems, a natural approach is to use Lyapunov-based theory and Variable Structure Systems theory. The class of controls (to be synthesized) is such that the controls overcome uncertainties and disturbances for a limited class of nonlinear systems, described in chapters 2-5.

1.5 Notation and mathematical preliminaries

First some basic notation is introduced which is used throughout this thesis. The state space is denoted by $X := \mathbb{R}^n$ and the control space by $U := \mathbb{R}^m$, where $1 \leq m \leq n$. The Euclidean inner product (on X or U as appropriate) and the induced norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. For a linear map L, $\|L\| = \{\max \sigma(L^TL)\}^{\frac{1}{2}}$, where σ denotes spectrum, and no distinction is made between a map L and its matrix representation. Ker(L) and im(L) denote the kernel (or null space) and image of the map L, respectively. The "distance" between a point $a \notin B \subset X$ and the compact set B is defined to be

$$d(a,B) := \inf\{\ \|a-b\|\ :\ b \in B\}.$$

For a compact subset $A \neq \emptyset$ of X, ∂A denotes the boundary of A, and, for $\varepsilon > 0$, $N(A,\varepsilon)$ denotes the ε-neighbourhood of A defined by

$$N(A,\varepsilon) := \{x \in X\colon d(x,\ A) < \varepsilon\}.$$

For $x \in X$ and $S_1,\ S_2 \subset X$,

$$x + S_1 := \{x + s_1\colon s_1 \in S_1\},$$

$$S_1 + S_2 := \{s_1 + s_2\colon s_1 \in S_1,\ s_2 \in S_2\},$$

and

$$\langle\langle x,\ S_1\rangle\rangle := \{\langle x,\ s_1\rangle\colon s_1 \in S_1\} \subset \mathbb{R}.$$

Also, for a compact subset $K \neq \emptyset$ of X or U, $\xi(K) := \max\{\ \|v\|\colon v \in K\}$ and $\xi(\emptyset) := 0$. For a subspace $S \subset X$, Π_S denotes the orthogonal projector onto S. Finally B_X denotes the open unit ball in X, with closure $\bar{B}_X$.

Some definitions and properties relating to convex sets are now stated.

Defn.1.5.1: A subset K of X is said to be *convex* if, given $x_1,\ x_2 \in K$,

$$\alpha x_1 + (1-\alpha)x_2 \in K \text{ for all } 0 \leq \alpha \leq 1.$$

Remark: $\bar{B}_X$ is convex.

Proposition 1.5.1:(*i*) The sum of two convex sets is convex.

(*ii*) The image of a convex set under a linear mapping is convex.

(*iii*)The intersection of any family of convex sets is convex.

Proof:See Rockafellar [61], theorems 2.1, 3.1 and 3.4.

□

Defn.1.5.2:The ***convex hull*** of $K \subset X$, denoted by co(K), is the smallest convex set containing K.

The closed convex hull of K, denoted by $\overline{co}(K)$, is defined by $\overline{co}(K) := \overline{co(K)}$.

General properties and definitions relating to continuity and boundedness of functions now follow:

Defn.1.5.3:A function $f: [\alpha,\beta] \to X$ is ***absolutely continuous*** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum |f(\beta_i) - f(\alpha_i)| < \varepsilon$$

for any finite collection of non-overlapping subintervals $[\alpha_i, \beta_i] \subset [\alpha,\beta]$ which satisfy

$$\sum (\beta_i - \alpha_i) < \delta.$$

Remark: Every absolutely continuous function, f, is continuous and of bounded variation. If f is an absolutely continuous function then its derivative exists almost everywhere (except on a set of Lebesgue measure zero) and f is the integral of its derivative.

Defn.1.5.4: Suppose f: $S \to \mathbb{R}$, $S \subset X$ and $x_0 \in S$. Then f is ***upper semicontinuous at x_0*** iff, given $\varepsilon>0$, $\exists$ a neighbourhood N of x_0 for which $f(x) < f(x_0) + \varepsilon$ for all $x \in N \cap S$;

if f is upper semicontinuous at every $x \in S$ then f is said to be ***upper semicontinuous on S***.

Remark: An equivalent definition of upper semicontinuity is that

$f(\cdot)$ is upper semicontinuous at x_0 if $\limsup\limits_{x\to x_0} f(x) \leq f(x_0)$.

Proposition 1.5.2: If g: $\mathbb{R} \to \mathbb{R}$ is upper semicontinuous and bounded above

$$\text{then} \quad f: t \to \int_a^t g(s)\,ds, \quad t > a, \quad \text{is continuous.}$$

Proof: Since g is upper semicontinuous and bounded above, there exist a constant $0 < M < \infty$ and a sequence of continuous functions $\{\psi_n(t);\ n \in \mathbb{N}\}$ such that

$$M \geq \psi_1(t) \geq \psi_2(t) \geq \ldots.., \quad \forall\ t \in \mathbb{R},$$

and $\lim\limits_{n\to\infty} \psi_n(t) = g(t)$ for all $t \in \mathbb{R}$ (McShane [57], chapter 1, §7.9).

Given $\varepsilon>0$, let $|t - \tau| < \delta$, where $\delta := \varepsilon M^{-1}$, then

$$\begin{aligned}
|f(t) - f(\tau)| &= \left| \int_\tau^t g(s)\,ds \right| \\
&= \left| \int_\tau^t \lim_{n\to\infty} \psi_n(s)\,ds \right| \\
&= \left| \lim_{n\to\infty} \int_\tau^t \psi_n(s)\,ds \right|, \quad \text{since } \psi_n \text{ are continuous and converge uniformly to g on } \mathbb{R}, \\
&\leq M|t - \tau| \\
&< \varepsilon.
\end{aligned}$$

□

Defn. 1.5.5: A function $h: R \times Y \to R^n$, where Y may be $R^n \times R^p$, or $R^n \times R^n \times R^p$ (as appropriate), is ***Carathéodory*** iff :

(*i*) $h(\cdot,y)$ is Lebesgue measurable for each $y \in Y$;

(*ii*) $h(t,\cdot)$ is continuous for each $t \in R$;

(*iii*) for each compact set $U \subset R \times Y$, there exists a Lebesgue integrable function $m_U(\cdot)$ such that

$$\|h(t,y)\| \leq m_U(t)$$

for all $(t,y) \in U$.

If, in addition, $m_U(\cdot) \equiv m_U$, a constant, then h is said to be ***strongly Carathéodory***.

Defn. 1.5.6: If a function is Lebesgue measureable and bounded almost everywhere (a.e.), then the function is said to be ***essentially bounded***.

$L^\infty(R;R^p)$ denotes the set of all essentially bounded functions defined on R with values in R^p; $C(E)$ and $C^1(E)$ denote the set of all real functions, defined on $E \subset X$, which are continuous and continuously differentiable, respectively.

General properties and definitions are now introduced, relating to continuity and compactness of multifunctions.

Let Y_1 and Y_2 be Hausdorff topological spaces. A multifunction $\mathcal{Y}: Y_1 \rightrightarrows Y_2$, $y \mapsto \mathcal{Y}(y) \subset Y_2$, is a mapping of Y_1 into the subsets of Y_2.

Defn. 1.5.7: $\mathcal{Y}$ (with non-empty values) is said to be ***upper semicontinuous at*** $y_1 \in Y_1$ if, for each open set $N_2 \supset \mathcal{Y}(y_1)$, there exists a neighbourhood N_1 of y_1 such that $\mathcal{Y}(N_1) \subset N_2$.

$\mathcal{Y}$ is said to be ***upper semicontinuous*** if it is upper semicontinuous at each $y_1 \in Y_1$.

If, as in this thesis, Y_1 and Y_2 are real Banach spaces and $\mathcal{Y}: Y_1 \rightrightarrows Y_2$ has **compact** values, then the above definition of upper semicontinuity of $\mathcal{Y}$ at $y_1 \in Y_1$ is equivalent to the following : for each $\varepsilon > 0$, $\exists\ \delta > 0$ such that $\mathcal{Y}(y) \subset \mathcal{Y}(y_1) + \varepsilon B_{Y_2}$, for all $y \in y_1 + \delta B_{Y_1}$.

The following properties will be invoked later.

Proposition 1.5.3: If D is any closed set, $\mathcal{Y}: D \rightrightarrows Y_2$ has closed values and $\overline{\mathcal{Y}(D)}$ is compact, then $\mathcal{Y}: D \rightrightarrows Y_2$ is upper semicontinuous iff the graph of $\mathcal{Y}$ is closed.

Proof: This is a consequence of proposition 2 and corollary 1, §1, chapter 1, Aubin and Cellina [7].

□

The above proposition is useful in determining whether a multifunction is upper semicontinuous or not. For example, the multifunction $\mathcal{Y}_1: \mathbb{R} \rightrightarrows \mathbb{R}$, defined by

$$\mathcal{Y}_1(y) := \begin{cases} [-1,1], & y=0 \\ \{0\}, & y \neq 0 \end{cases}$$

is upper semicontinuous, since $\overline{\mathcal{Y}_1(\mathbb{R})} = [-1,1]$ and the graph of $\mathcal{Y}_1$ is closed. The multifunction $\mathcal{Y}_2: \mathbb{R} \rightrightarrows \mathbb{R}$, defined by

$$\mathcal{Y}_2(y) := \begin{cases} \{0\}, & y=0 \\ [-1,1], & y \neq 0 \end{cases}$$

is not upper semicontinuous. In this case, $\overline{\mathcal{Y}_2(\mathbb{R})}$ is the compact set

$[-1,1]$, but the graph of $\mathcal{Y}_2$ is not closed since there exists a sequence $\{y_n \in \mathbb{R};\ n \in \mathbb{N}\}$ such that $y_n \to 0$, $\mathcal{Y}_2(y_n) \to \frac{1}{2}$ (say) as $n \to \infty$ but $(0,\frac{1}{2})$ does not belong to the graph of $\mathcal{Y}_2$.

<u>Proposition 1.5.4</u>:Let $\mathcal{Y}_1$: $Y_1 \rightrightarrows Y_2$ and $\mathcal{Y}_2$: $Y_2 \rightrightarrows Y_3$ have non-empty values. If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are upper semicontinuous, then $\mathcal{Y}_2 o \mathcal{Y}_1$ is upper semicontinuous, where $\mathcal{Y}_2 o \mathcal{Y}_1$: $Y_1 \rightrightarrows Y_3$ is defined by $y \mapsto (\mathcal{Y}_2 o \mathcal{Y}_1)(y) := \bigcup_{\upsilon \in \mathcal{Y}_1(y)} \mathcal{Y}_2(\upsilon)$.

<u>Proof</u>: See Aubin and Cellina [7], chapter 1, §1, proposition 1.

□

<u>Proposition 1.5.5</u>:Suppose $K \subset Y_1$ is compact. If $\mathcal{Y}$: $Y_1 \rightrightarrows Y_2$ is upper semicontinuous with compact values, then $\mathcal{Y}(K) \subset Y_2$ is compact.

<u>Proof</u>: See Aubin and Cellina [7], chapter 1, §1, proposition 3.

□

<u>Proposition 1.5.6</u>:If f: $Y_1 \to \mathbb{R}$ is continuous and $\mathcal{Y}$: $Y_1 \rightrightarrows Y_2$ is upper semicontinuous with compact values, then $f\mathcal{Y}$: $Y_1 \rightrightarrows Y_2$, $y \mapsto f(y)\mathcal{Y}(y)$, is upper semicontinuous with compact values.

<u>Proof</u>:Let $y_1 \in Y_1$ and suppose $[\xi(\mathcal{Y}(y_1))]^{-1} \neq 0$. For $\varepsilon > 0$, define

$$\varepsilon_2 := \tfrac{1}{2}\varepsilon[\xi(\mathcal{Y}(y_1))]^{-1} \quad \text{and} \quad \varepsilon_1 := \tfrac{1}{2}\varepsilon[|f(y_1)| + \varepsilon_2]^{-1}.$$

Since $\mathcal{Y}$ is upper semicontinuous at y_1, there exists $\delta_1(\varepsilon_1) > 0$ such that

$$\mathcal{Y}(y) \subset \mathcal{Y}(y_1) + \varepsilon_1 \mathcal{B}_{Y_2} \quad \text{for all } y \in y_1 + \delta_1 \mathcal{B}_{Y_1}.$$

Since f is continuous at y_1, $\exists\ \delta_2(\epsilon_2) > 0$ such that

$$|f(y) - f(y_1)| < \epsilon_2$$

for all $y \in y_1 + \delta_2 B_{Y_1}$.

Now define $\delta(\epsilon) := \min\{\delta_1, \delta_2\}$, then,

for all $y \in y_1 + \delta B_{Y_1}$,

$$\begin{aligned} f(y)Y(y) &\subset f(y)Y(y_1) + \epsilon_1 |f(y)| B_{Y_2} \\ &\subset f(y_1)Y(y_1) + \epsilon_2 \xi(Y(y_1)) B_{Y_2} + \epsilon_1 |f(y)| B_{Y_2}. \\ &\subset f(y_1)Y(y_1) + \tfrac{1}{2}\epsilon B_{Y_2} + \epsilon_1[|f(y_1)| + \epsilon_2] B_{Y_2}, \\ &\subset f(y_1)Y(y_1) + \epsilon B_{Y_2} . \end{aligned}$$

Note that, when $\xi(Y(y_1)) = 0$, the above inclusion is still satisfied, with $\epsilon_1 = \frac{1}{2}\epsilon[|f(y_1)| + \epsilon_2]^{-1}$. Since f takes values in $\mathbb{R}$ and Y has compact values, fY has compact values and hence fY is upper semicontinuous at y_1. Since $y_1 \in Y_1$ is arbitrary, $fY: Y_1 \rightrightarrows Y_2$ is upper semicontinuous.

□

Some notation and results pertaining to real matrices now follows. $\mathbb{R}^{m\times n}$ denotes the set of real matrices of order mxn and a positive definite $P \in \mathbb{R}^{n\times n}$ is denoted by $P > 0$. Also, $\sigma_{min}(\cdot)$ and $\sigma_{max}(\cdot)$ denote the respective minimum and maximum eigenvalues of a real, symmetric matrix. For any two symmetric $P, Q \in \mathbb{R}^{n\times n}$, satisfying $P, Q > 0$, Kalman and Bertram [42] prove the following :

Proposition 1.5.7:For any $x \in X$ and symmetric $P, Q \in \mathbb{R}^{n\times n}$, satisfying

$$P, Q > 0,$$

$$\sigma_{min}(P^{-1}Q)\langle x, Px\rangle \leq \langle x, Qx\rangle \leq \sigma_{max}(P^{-1}Q)\langle x, Px\rangle .$$

Noting that if $P > 0$ then $P^{-1} > 0$, the following elementary proposition can be stated :

Proposition 1.5.8:If $P > 0$ then $\sigma_{max}(P) = \{\sigma_{min}(P^{-1})\}^{-1}$

and $\sigma_{min}(P) = \{\sigma_{max}(P^{-1})\}^{-1}$.

Proposition 1.5.9:Suppose $A, P \in \mathbb{R}^{nxn}$ are real and satisfy $PA + A^TP = 0$, where $P > 0$ is symmetric, then elements of $\sigma(A)$ are either zero or purely imaginary.

Proof:Since P is real and symmetric, there exists a real, orthogonal $Q \in \mathbb{R}^{nxn}$ such that $P = Q^T\Lambda Q$, where Λ is diagonal and $\lambda_i \in \sigma(P)$ are the diagonal elements. Let * denote complex conjugate, then, for any $x \in \mathbb{C}^n$,

$$
\begin{aligned}
(x^*)^T P x &= (x^*)^T Q^T \Lambda Q x \\
&= (y^*)^T \Lambda y \,, \qquad \text{where } y=[y_1\ y_2\ \ldots.y_n]^T = Qx, \\
&= \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \ldots. + \lambda_n |y_n|^2 \\
&> 0 \,, \qquad \text{since } P > 0.
\end{aligned}
$$

Let $\mu(\neq 0) \in \sigma(A)$ and x the corresponding eigenvector.

Then

$$
\begin{aligned}
\mu^*(x^*)^T P x &= [\mu(x^*)^T P x]^* \,, \qquad \text{since } (x^*)^T P x \text{ is real,} \\
&= x^T P A x^* \,, \qquad \text{since } \mu \in \sigma(A) \text{ and } PA \text{ is real.}
\end{aligned}
$$

Hence
$$
\begin{aligned}
\mu^*(x^*)^T P x &= (x^T P A x^*)^T \\
&= -(x^*)^T P A x \,, \qquad \text{since } PA + A^T P = 0, \\
&= -\mu(x^*)^T P x \,,
\end{aligned}
$$

i.e. $(\mu^* + \mu)(x^*)^T P x = 0$.

But $(x^*)^T P x > 0 \Rightarrow \mu^* + \mu = 0$, i.e. μ is purely imaginary.

Note that if $\mu x = Ax$ then $\mu^* x^* = Ax^* \Rightarrow \mu^* \in \sigma(A)$.

□

Finally, ***Zorn's lemma*** is stated, which is used in establishing maximal intervals of existence for solutions of differential inclusions (see chapter 3).

Proposition 1.5.10: Let $\Upsilon \neq \emptyset$ be a partially ordered set. Suppose that every totally ordered subset $\Phi \subset \Upsilon$ has an upper bound. Then Υ has at least one maximal element.

2.DETERMINISTIC CONTROL OF UNCERTAIN SYSTEMS MODELLED BY DIFFERENTIAL EQUATIONS

2.1 Introduction

The design of stabilizing nonlinear continuous state feedback controls for a class of uncertain dynamical systems is considered. The uncertain systems, modelled by nonlinear differential equations, consist of a known nominal linear system together with an uncertain nonlinear perturbation from a class of perturbations which encompass all possible realizations of uncertainty. The class of perturbations is assumed to be known in the sense that it is characterized by known functional properties and bounds. The approach is based on the deterministic theory of feedback control in the presence of uncertainty developed in, for example, Barmish, Corless and Leitmann [9], Barmish and Leitmann [10], Corless and Leitmann [21], Corless, Leitmann and Ryan [22], Gutman [31], Gutman and Leitman [32], Leitmann [50], Ryan and Corless [67], Ryan, Leitmann and Corless [68].

Before any stability analysis can be undertaken, stability concepts must be defined and stability criteria derived. Many different kinds of stability for dynamical systems have been investigated (see, for example, Kalman and Bertram [42], LaSalle and Lefschetz [47], Massera [56], Sell [73], Willems [84]; a good survey being Antosiewicz [4]). More recently Walker [82], in particular, has contributed to this area of study. Here, in this thesis, only a few types of stability will be considered. For instance, stability properties with respect to some set will be required. This topic has been discussed by, amongst others, Bhatia and Szego [12], LaSalle [46], Lefschetz [48], Roxin [62], Yoshizawa [87] and Zubov [94]. The appropriate stability concepts are defined in §2.2.

The stability criteria, derived in §2.3, consist of a set of (sufficient) conditions. These conditions comprise Lyapunov's (second) method and when satisfied the method may be invoked to deduce certain stability results relating to particular dynamical systems. The differential equations modelling the class of uncertain dynamical systems being considered are described in §2.4. Based on concepts from both Lyapunov theory and the work by Leitmann *et al.*, a class C of continuous feedback controls is proposed (in §2.5) which renders the zero state of the state equations representing the uncertain dynamical systems "practically" stable in the sense that, given any compact set containing the zero state, there exists a control in C which guarantees global uniform finite-time stability (defined in §2.2) of the prescribed compact set. Finally, in §2.6 and §2.7, the approach is extended to problems of ***model-following*** and ***tracking*** of uncertain dynamical systems; a model-following example being provided in §2.8.

A survey of some model-following techniques is given by Landau [44]. The basic idea of a model-following control system is the use of a ***reference model***, which specifies the design objectives as a part of the control system. The objective is then to minimize the error between the states of the model and the reference model. For such systems a class of continuous feedback controls, robust with respect to the uncertainty in the system, is designed which guarantees that the error remains bounded and, in addition, tends to a calculable neighbourhood of the error-state origin. This can be made arbitrarily small by appropriate choice of the control. The restriction to continuous feedback controls can be relaxed to allow for discontinuous feedback controls (see chapter 4). In this case, the differential equations modelling the system are replaced by a ***differential inclusion*** and it can be shown that "perfect" model-following in the presence of uncertainty may be obtained, i.e. for the error system, it can be shown that the error-state origin is globally asymptotically stable.

The problem of tracking in the presence of uncertainties has been investigated by Ambrosino, Celentano and Garofalo [2], [3], Corless, Leitmann and Ryan [22], Ryan, Leitmann and Corless [68], Slotine and Sastry [76]. For this problem, a class of continuous feedback controls is synthesized such that, given a feasible path to be tracked and an arbitrary small neighbourhood of the origin in the appropriate error space, the tracking error for the feedback controlled uncertain system is ultimately bounded with respect to the prescribed neighbourhood. These feedback controls are robust with respect to disturbance signals and parameter variations. Discontinuous feedback controls can be used to obtain "perfect" tracking in the presence of disturbances and parameter variations.

2.2 Stability concepts

Consider the ordinary differential equation

$$\dot{x}(t) = f(t,x(t)) \;, \qquad (2.2.1a)$$

$$x(t_0) = x_0 \;, \qquad (2.2.1b)$$

where f: $\mathbb{R}\times X \to X$. Concepts relating to a solution of (2.2.1) are now defined.

Defn.2.2.1: A ***local solution*** of (2.2.1) is any absolutely continuous function x: $[t_0,t_1) \to X$ satisfying (2.2.1a) almost everywhere and (2.2.1b); if $t_1=\infty$ then $t \mapsto x(t)$ is said to be a ***global solution*** of (2.2.1).

Defn.2.2.2: The differential equation (2.2.1) has ***existence*** of local solutions iff, given any pair $(t_0,x_0) \in \mathbb{R}\times X$, there exists a local solution x: $[t_0,t_1) \to X$ of (2.2.1).

Defn.2.2.3: Suppose x: $[t_0,t_1) \to X$ is a local solution of (2.2.1). The interval $[t_0,t_1)$ is a ***maximal interval of existence*** and the solution x is said to be ***maximal*** iff x does not have a proper extension which is also a solution.

The following proposition (see Hale [34]) yields sufficient conditions for the existence of a local solution which can be extended into a maximal solution.

<u>Proposition 2.2.1</u>:If f: $\mathbb{R}\times X \to X$ is a Carathéodory function, then, for any (t_0,x_0) in $\mathbb{R}\times X$, there exists a local solution of (2.2.1) which can be continued into a maximal solution.

<u>Defn.2.2.4</u>:The system (2.2.1) has ***indefinite continuation*** of solutions iff, for each $(t_0,x_0) \in \mathbb{R}\times X$, every solution x of (2.2.1) has maximal interval of existence $[t_0,\infty)$.

Useful sufficient conditions for indefinite continuation of solutions are given in the following proposition which may be deduced from the results presented in Hale [34], chapter 1.

<u>Proposition 2.2.2</u>:Suppose f: $\mathbb{R}\times X \to X$ is a Carathéodory function. If, for each $(t_0,x_0) \in \mathbb{R}\times X$, every maximal solution x is bounded, then system (2.2.1) has indefinite continuation of solutions.

For the remaining part of this section it is assumed that f is a Carathéodory function.

A system represented by (2.2.1) is required to exhibit "desirable" dynamic behaviour described in terms of the properties: boundedness and stability. There now follows formal definitions relating to the above two concepts .

<u>Defn.2.2.5</u>:A system modelled by (2.2.1) has ***global uniform boundedness*** of solutions iff it has indefinite continuation of solutions and, for each $\rho > 0$, there exists $r(\rho) > 0$ (independent of t_0) such that

$x(t) \in r(\rho)\overline{B}_X$ for all $t \geq t_0$ on every solution $x(\cdot)$ of (2.2.1) with $x_0 \in \rho\overline{B}_X$.

Definitions relating to a nonempty set M, exhibiting desirable stability properties are stated below:

Defn.2.2.6:M $\subset$ X is a ***uniformly stable*** set for system (2.2.1) iff, for each $\varepsilon > 0$ and $t_0 \in R$, there exists $\delta(\varepsilon) > 0$ (independent of t_0) such that if $x_0 \in N(M,\delta(\varepsilon))$ then, for every maximal solution x of (2.2.1), $x(t) \in N(M,\varepsilon)$ for all $t \geq t_0$.

Defn.2.2.7:M $\subset$ X is a ***uniformly weakly attractive*** set for system (2.2.1) iff, there exists $\delta > 0$ and to each $\varepsilon > 0$ there corresponds a number $T_\delta(\varepsilon) > 0$ (independent of t_0) such that if $x_0 \in N(M,\delta)$ then, for every maximal solution x of (2.2.1), $x(t) \in N(M,\varepsilon)$ for all $t \geq t_0+T_\delta(\varepsilon)$; if (2.2.1) has indefinite continuation of solutions and the above conditions hold with δ arbitrarily large, then M is said to be ***globally uniformly weakly attractive*** for system (2.2.1).

Defn.2.2.8:M $\subset$ X is ***uniformly asymptotically stable*** for system (2.2.1) iff M is both uniformly stable and uniformly weakly attractive; if (*i*) system (2.2.1) has indefinite continuation of solutions and global uniform boundedness of solutions, (*ii*) M is uniformly stable, (*iii*) M is globally uniformly weakly attractive, then M is said to be ***globally uniformly asymptotically stable*** for system (2.2.1).

Defn.2.2.9:M $\subset$ X is a ***uniformly strongly attractive*** set for system (2.2.1) iff there exists $\delta > 0$ and $\tau(\delta) > 0$ (independent of t_0) such that if

$x_0 \in N(\mathrm{M},\delta)$, then for every maximal solution x of (2.2.1), $x(\mathrm{t}) \in \mathrm{M}$ for all $\mathrm{t} \geq \mathrm{t}_0+\tau(\delta)$; if (2.2.1) has indefinite continuation of solutions and the above conditions hold with δ arbitrarily large, then M is said to be ***globally uniformly strongly attractive*** for system (2.2.1).

Defn.2.2.10:M $\subset$ X is ***globally uniformly finite–time stable*** for system (2.2.1) iff (*i*) (2.2.1) has indefinite continuation of solutions and global uniform boundedness of solutions, (*ii*) M is uniformly stable, (*iii*) M is globally uniformly strongly attractive.

There are two main reasons why the previous definitions relating to stability of sets are given. The basic problem to be addressed in this thesis is to obtain stabilizing feedback controllers for uncertain dynamical systems. The design of these controllers is based on Lyapunov theory which utilizes the notion of a "Lyapunov function". One of the conditions sufficient for stability is that the time derivative of the "Lyapunov function" is negative when evaluated along all solutions. When considering stability of the state origin it is often the case that the above condition is only satisfied for solutions contained in the complement of a known compact set containing the state origin. Hence, in this case stability of sets must be considered. A second reason is that variable structure system theory is invoked, of which a basic concept is the notion of an attractive manifold. Thus, when considering global stability , one must consider attractivity of a set (i.e. the manifold in state space).

2.3 Lyapunov functions and stability criteria

"Lyapunov functions" play an important role in investigating stability properties of dynamical systems using Lyapunov's method. The Lyapunov approach is to show that the "Lyapunov function" is nonincreasing along all solutions to (2.2.1) by means that do not require explicit knowledge of solutions to (2.2.1). From this, appropriate conclusions may be drawn regarding stability concepts relating to solutions of the differential equation (2.2.1). An essential part of Lyapunov's method is the determination of the time derivative of the "Lyapunov function" along all solutions of the dynamical system. Consider a Lyapunov candidate $(t,x) \mapsto V(t,x)$: $\mathbb{R}\times X \to \mathbb{R}$ which satisfies the smoothness condition $V \in C^1(\mathbb{R}\times X)$, i.e. V has continuous partial derivatives of the first order, in which case its time derivative along solutions to (2.2.1) is given by

$$\frac{\partial V}{\partial t}(t,x(t)) + \langle \mathrm{grad}\ V(t,x(t)),\ f(t,x(t))\rangle, \qquad \text{for almost all } t \in \mathbb{R}.$$

There are many instances when candidate "Lyapunov functions" are not differentiable and therefore the ensuing analysis is developed for a nonsmooth function V: $\mathbb{R}\times X \to \mathbb{R}$, specifically, V is assumed continuous but not necessarily differentiable. For the remaining part of this chapter, it is assumed that V satisfies a local Lipschitz condition (which implies continuity but not necessarily differentiability).

Defn.2.3.1: A function f: $\mathbb{R}\times X \to \mathbb{R}$ is said to be ***locally Lipschitz*** if, for each $(t,x) \in \mathbb{R}\times X$, there exists $k > 0$ and $\varepsilon > 0$ such that

$$|f(\tau_1,y) - f(\tau_2,z)| < k\|(\tau_1,y) - (\tau_2,z)\|,$$

for all $(\tau_1,y),\ (\tau_2,z) \in (t,x) + \varepsilon\bar{B}_{\mathbb{R}\times X}$.

Lyapunov stability results, using locally Lipschitz functions, have been investigated by Antosiewicz [4], LaSalle [46], Massera [56], and Yoshizawa [87] (to name but a few). In this case, the time derivative of $t \mapsto V(t,x(t))$ must be interpreted in a generalized sense. Let

$$DV(t,x(t)) := \liminf_{h\downarrow 0} \frac{V(t+h,x(t+h)) - V(t,x(t))}{h}$$

denote the lower right *Dini* derivative of $V(\cdot,x(\cdot))$ at t. As mentioned earlier, the Lyapunov approach requires the evaluation of $DV(t,x(t))$ a.e. along solutions $x(\cdot)$ of (2.2.1) without explicit knowledge of these solutions. This is achieved, in part, by introducing the (lower right) directional derivative of V at $(t,x) \in \mathbb{R}\times X$ in the direction $w \in X$:

$$D_+V(t,x;w) := \liminf_{h\downarrow 0} \frac{V(t+h,x+hw) - V(t,x)}{h}$$

and using the following proposition:

Proposition 2.3.1: Suppose $V: \mathbb{R}\times X \to \mathbb{R}$ is locally Lipschitz and f is a Carathéodory function, then, along all solutions of (2.2.1), $DV(t,x(t)) = D_+V(t,x(t);f(t,x(t)))$ for almost all t.

Proof: (See Yoshizawa [87].)

Let $x: [t_0,t_1) \to X$ be a maximal solution of (2.2.1). Since f is Carathéodory, such a solution exists, by proposition 2.2.1. Let $Z(x) \subset \mathbb{R}$ denote the set of measure zero on which the derivative $\dot{x}(t)$ fails to exist. Since V is locally Lipschitz, for each $t \in \mathbb{R}^+\setminus Z(x)$, there exists a Lipschitz constant k such that, for h sufficiently small,

$$\begin{aligned} &V(t+h,x(t+h)) - V(t,x(t)) \\ &= V(t+h,x(t+h)) - V(t+h,x(t)+h\dot{x}(t)) + V(t+h,x(t)+h\dot{x}(t)) - V(t,x(t)) \\ &\leq k\|(t+h,x(t+h)-x(t)-h\dot{x}(t))\| + V(t+h,x(t)+h\dot{x}(t)) - V(t,x(t)). \end{aligned}$$

Hence, along the solution $x(\cdot)$, it follows that

$$DV(t,x(t)) \leq D_+V(t,x(t);f(t,x(t))) \quad \forall t \in [t_0,t_1)\setminus Z(x).$$

Similarly, for h sufficiently small,

$$V(t+h,x(t)+h\dot{x}(t)) - V(t,x(t))$$

$$= V(t+h,x(t)+h\dot{x}(t)) - V(t+h,x(t+h)) + V(t+h,x(t+h)) - V(t,x(t))$$

$$\leq k\|(t+h,x(t)+h\dot{x}(t)-x(t+h))\| + V(t+h,x(t+h)) - V(t,x(t))$$

which implies that, along the solution $x(\cdot)$,

$$D_+V(t,x(t);f(t,x(t))) \leq DV(t,x(t)) \quad \forall t \in [t_0,t_1)\backslash Z(x).$$

□

<u>Proposition 2.3.2</u>: Suppose $\tilde{V}: R \to R$ is continuous and $W: R \to R$ is an upper semicontinuous function which is bounded above.

If $D\tilde{V}(t) + W(t) \leq 0$ a.e., then

$$\tilde{V}(b) \leq \tilde{V}(a) - \int_a^b W(s)\, ds \qquad \text{for all } a < b.$$

<u>Proof</u>: (See Aubin and Cellina [7], chap.6, §1, proposition 8.)

Suppose that for some $\varepsilon > 0$,

$$\tilde{V}(b) - \tilde{V}(a) + \int_a^b W(s)\, ds > \varepsilon(b-a).$$

Define $t \mapsto g(t) := \tilde{V}(t) - \tilde{V}(a) + \int_a^t W(s)\, ds - \varepsilon(t-a)$,

then, as a consequence of proposition 1.5.2, g is continuous. Also,

$$Dg(t) = \liminf_{h \downarrow 0} \left\{ \frac{\tilde{V}(t+h) - \tilde{V}(t)}{h} + \frac{1}{h}\int_t^{t+h} W(\tau)\, d\tau - \varepsilon \right\}$$

Since W is upper semicontinuous, it is possible to fix $\eta > 0$ and choose $\delta > 0$ such that $0 < \tau - t < \delta$ implies that $W(t) + \eta \geq W(\tau)$.

Hence $$Dg(t) \leqslant D\tilde{V}(t) + W(t) + \eta - \varepsilon$$

and, since η is arbitrary,

$$Dg(t) \leqslant D\tilde{V}(t) + W(t) - \varepsilon < 0.$$

In particular, $Dg(a) < 0$.

Since $g(a)= 0$, then, for some t_1 near a with $t_1 > a$, $g(t_1) < 0$. Since $g(b) > 0$ and g is continuous, there exists t_2, defined by

$$t_2 := \mathit{sup}\{t<b: g(t)= 0\},$$

which satisfies $a<t_1<t_2<b$. Hence, $g(t) > 0$ for $t \in (t_2,b)$ from which it can be inferred that $Dg(t_2) \geqslant 0$.

This contradicts $Dg(t) < 0$, $t \in (a,b)$.

□

Remark: In the above proposition, the requirement that the function W be bounded above is superfluous if W is continuous.

Stability criteria are now considered for some nonempty compact set $M \subset X$ (with boundary ∂M) with respect to the dynamical system (2.2.1), where it is assumed that f is a Carathéodory function. By proposition 2.2.1, the condition that f is Carathéodory is sufficient for the existence of a local solution. The existence of a function $(t,x) \mapsto V(t,x)$ satisfying prescribed conditions on $\mathbb{R}\times X$, ensures that the compact set M has important stability properties w.r.t. the differential equation (2.2.1).

Lemma 2.3.1: Let $M \subset X$ be compact and nonempty. Suppose there exists a locally Lipschitz function $V: \mathbb{R}\times X \to \mathbb{R}$ such that, for all $(t,x) \in \mathbb{R}\times X$, V satisfies:

(a) $V(t,x)= 0$ for all $(t,x) \in \mathbb{R}\times\partial M$

(b) there exists strictly increasing functions $W_1,W_2: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, with W_1, $W_2 \in C(R_0^+)$, such that $W_1(0)= W_2(0)= 0$ and

$W_2(d(x,M)) \geq V(t,x) \geq W_1(d(x,M))$, for all $(t,x) \in R\times(X\backslash M)$

(c) there exists either (*i*) a strictly increasing continuous function W_3: $R_0^+ \to R_0^+$, or (*ii*) a strictly increasing upper semicontinuous function W_3: $R_0^+ \to R_0^+$, bounded above, such that,

$$D_+V(t,x;f(t,x)) + W_3(d(x,M)) \leq 0 \tag{2.3.1}$$

for all $(t,x) \in R\times(X\backslash M)$,

(d) $W_1(d(x,M)) \to \infty$ as $d(x,M) \to \infty$, i.e. V is "radially unbounded",

then the set M is a globally uniformly asymptotically stable set for system (2.2.1).

Proof: First it is shown that (2.2.1) has indefinite continuation of solutions. Suppose x: $[t_0,t_1) \to X$ is a maximal solution and let $[\alpha_0,\alpha_1] \subset [t_0,t_1)$ be arbitrary. Then, by the continuity of x, there exists a compact $K \subset X$ such that $x(t) \in K$, $\forall\ t \in [\alpha_0,\alpha_1]$. Since V is locally Lipschitz, it is Lipschitz on the compact set $[\alpha_0,\alpha_1]\times K$. Therefore $t \mapsto V(t,x(t))$ restricted to $[\alpha_0,\alpha_1]$ is a composition of an absolutely continuous function x and Lipschitz function V and hence is absolutely continuous on $[\alpha_0,\alpha_1]$ (see McShane [57], chap.1, §9). It now follows from proposition 2.3.1 and the inequality (2.3.1) that

$$V(t,x(t))= V(\alpha_0,x(\alpha_0)) + \int_{\alpha_0}^{t} D_+V(s,x(s);f(s,x(s)))ds \leq V(\alpha_0,x(\alpha_0))$$

$\forall\ t \in [\alpha_0,\alpha_1]$, i.e. $t \mapsto V(t,x(t))$ is non-increasing on $[\alpha_0,\alpha_1]$. Since $[\alpha_0,\alpha_1]$ is arbitrary, $V \circ x$ is absolutely continuous and non-increasing on $[t_0,t_1)$. It will now be shown that $x(\cdot)$ is bounded on $[t_0,t_1)$. Let $\lambda > 0$

be such that $d(x_0,M) \leq \lambda$ and let $c(\lambda) > 0$ be such that $V(t_0,x_0) \leq c(\lambda)$ (a suitable choice is $c(\lambda) = W_2(\lambda)$). As a consequence of (b) and (d), there exists $\mu(\lambda) > \lambda$ such that $V(t,x) > c(\lambda)$ for some $t \in [t_0,t_1)$ with $d(x,M) = \mu$. The aim is now to prove that $d(x(t),M) < \mu \; \forall t \in [t_0,t_1)$. This can be achieved by the following contradiction argument.

Suppose $d(x(t),M) \geq \mu$ for some $t \in [t_0,t_1)$. Then $\exists \; \tau_1, \tau_2$, satisfying $t_0 \leq \tau_1 < \tau_2 \leq t_1$ such that

$$d(x(\tau_1),M) = \lambda, \quad d(x(\tau_2),M) = \mu \quad \text{and} \quad \lambda < d(x(t),M) < \mu,$$

$\forall \; t \in (\tau_1,\tau_2)$. Therefore $V(\tau_2,x(\tau_2)) > c(\lambda)$ and $V(\tau_1,x(\tau_1)) \leq c(\lambda)$ which contradicts the fact that $V \circ x$ is non-increasing on $[t_0,t_1)$. Hence $d(x(t),M) < \mu \quad \forall \; t \in [t_0,t_1)$ and, since M is compact, this implies that $\|x(\cdot)\|$ is bounded. One can now deduce, by proposition 2.2.2, that equation (2.2.1) has indefinite continuation of solutions.

By standard arguments (see, for example, Yoshizawa [87]), the compact set M can be shown to be uniformly stable as follows :

For any $t \in [t_0,\infty)$ and $d(x,M) > 0$, $\quad W_1(d(x,M)) \leq V(t,x) \leq W_2(d(x,M))$. Thus, given $\varepsilon > 0$ and taking $\delta = \delta(\varepsilon)$, independent of t_0, such that $W_1(\varepsilon) = W_2(\delta)$, then, since $t \mapsto V(t,x)(t)$ is nonincreasing along every solution $x(\cdot)$, it follows that, for $x_0 \in N(M,\delta)$,

$$W_1(d(x(t),M)) \leq V(t,x(t)) \leq V(t_0,x_0) \leq W_2(d(x_0,M)) < W_2(\delta) = W_1(\varepsilon).$$

Thus if $d(x_0,M) < \delta$, then $x(t) \in N(M,\varepsilon) \quad \forall \; t \geq t_0$.

Standard analysis (see, for example, Willems [84]) can be applied to show that all solutions x of (2.2.1), with trajectory in $X \backslash M$, are globally uniformly bounded and that M is globally weakly attractive as follows :

Since V is radially unbounded, for any $r > 0$ there exists $\rho > 0$ such that $W_1(\rho) > W_2(r)$. If $d(x_0,M) < r$ then $d(x(t),M) < \rho$ for $t \geq t_0$, since

$$W_1(\rho) > W_2(r) \geq V(t_0,x_0) \geq V(t,x(t)) \geq W_1(d(x(t),M))$$

and so all solutions $x \in X \backslash M$ of (2.1.1) are globally uniformly bounded. It remains to show that M is globally uniformly weakly attractive. Using propositions 2.3.1, 2.3.2 and inequality (2.3.1), one can conclude that, along solutions to (2.2.1),

$$V(t,x(t)) \leq V(t_0,x_0) - \int_{t_0}^{t} W_3(d(x(s),M))\, ds \tag{2.3.2}$$

under hypothesis (c) of this lemma. Given $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$, independent of t_0, such that $W_1(\varepsilon) = W_2(\delta)$.

Define $T := \dfrac{W_1(\rho)}{W_3(\delta)}$, then, since ρ is independent of t_0, T is independent of t_0 but depends on r and ε.

Whenever $d(x_0,M) < r$, $d(x_1,M) < \delta$ for some $t_1 \in [t_0,t_0+T]$, where $x_1 = x(t_1)$, since otherwise, if $d(x(t),M) \geq \delta \quad \forall t \in [t_0,t_0+T]$ then it immediately follows, from (2.3.2), that

$$\begin{aligned} V(t,x(t)) &\leq V(t_0,x_0) - (t-t_0)W_3(\delta) \\ &\leq W_2(d(x_0,M)) - (t-t_0)W_3(\delta) \\ &< W_2(r) - (t-t_0)W_3(\delta) \end{aligned}$$

which contradicts (b) when $t = t_0+T$.

For $t \geq t_1$, $d(x(t),M) < \delta$ and hence, along solutions,

$W_1(d(x(t),M)) \leq V(t,x(t)) \leq V(t_1,x_1) \leq W_2(d(x_1,M)) < W_2(\delta) = W_1(\varepsilon)$,

from which it is deduced that $d(x(t),M) < \varepsilon$. Hence a solution cannot leave $N(M,\varepsilon)$ for $t \geq t_1$ and, *a fortiori*, $t \geq t_0+T$. Thus, for $d(x_0,M) < r$, given $\varepsilon > 0$ $\exists\ T(r,\varepsilon) > 0$ (independent of t_0) such that, for each $t_0 \in \mathbb{R}$, a solution satisfies $x(t) \in N(M,\varepsilon) \quad \forall\ t \geq t_0 + T(r,\varepsilon)$. Since r is arbitrary, this proves that M is globally uniformly weakly attractive.

□

Remarks: (*i*) The behaviour of $V(t,x)$ in the interior of M is unimportant. It can be assumed that $V(t,x) = 0$ for all $(t,x) \in [t_0,\infty)\times M$.

(*ii*) Yoshizawa [87] (Chapter 4) examined the stability of a time-dependent set $M(t)$ under the additional hypothesis: for all (t,x), $(\tilde{t},x)$ belonging to a compact set Q in $\mathbb{R}\times X$, there exists a $\gamma > 0$, depending on Q, such that $| \, d(x,M(t)) - d(x,M(\tilde{t})) \, | \leq \gamma | \, t-\tilde{t} \, |$.

(*iii*) Any function V satisfying the conditions of lemmas 2.3.1 is termed a "Lyapunov function" for a dynamical system modelled by the differential equation (2.2.1).

2.4 Differential equation system model

Uncertain dynamical control systems modelled by differential equations of the form :

$$\dot{x}(t) = f(t,x(t),u(t)) \;, \quad x(t) \in X, \qquad u(t) \in U,$$

subject to $x(t_0) = x_0$, are considered. The function $f: \mathbb{R}\times X\times U \to X$ is unknown but partially identified in the sense that there exists a known pair (A,B) (where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ define a linear control system, henceforth referred to as the ***nominal*** linear system) such that

$$f(t,x,u) = Ax + Bu + g(t,x,u) \;, \text{ for all } (t,x,u),$$

and the unknown function g belongs to a known class *G* which comprises all possible uncertainties in the system description, together with any known time dependent or nonlinear elements. The uncertainty set *G* is now implicitly defined by the following assumptions:

A2.1 : For each $g \in G$, there exist Carathéodory functions g_1: $R \times X \to \ker(B^T)$ and g_2: $R \times X \times U \to U$ and a real (known) constant $\kappa_0 < 1$ such that

(*i*) $\Pi_{\ker(B^T)}\, g(t,x,u) = g_1(t,x)$, for all $(t,x,u) \in R \times X \times U$,

(*ii*) $\Pi_{\mathrm{im}(B)}\, g(t,x,u) = Bg_2(t,x,u)$, $\forall\ (t,x,u)$,

where B is assumed to have full rank m ($\leq$ n),

$\|g_2(t,x,u)\| \leq \alpha(t,x) + \kappa_0\|u\|$, for all $(t,x,u) \in R \times X \times U$, and

α: $R \times X \to R_0^+$ is a known Carathéodory function.

<u>Remark</u>: Assumption A2.1 indicates that, for all uncertainty realizations, the control can only directly influence the state on im(B), i.e. uncertainty represented by g_2 lies in the range of the control input. In the terminology of Barmish, Corless and Leitmann [9], Barmish and Leitmann [10], the function g_2 models the ***matched*** uncertainty in the system, while the function g_1 models the ***unmatched*** or ***residual*** uncertainty.

<u>2.5 Feedback stablization</u>

The control objective is to determine a Carathéodory feedback function u^*: $R \times X \to U$ such that, for arbitrary $g \in G$, a calculable compact set W, containing the state origin and "acceptably" small, is globally uniformly finite-time stable for the feedback system

$$\dot{x}(t) = Ax(t) + Bu^*(t,x(t)) + g(t,x(t),u^*(t,x(t))), \quad x(t_0) = x_0. \quad (2.5.1)$$

Following the approach taken by Leitmann *et al.*, the proposed feedback control, which is continuous in state, consists of a linear and nonlinear component. The linear component is chosen to stabilize the nominal linear system. The nonlinear component of the control, specified later, is constructed so that it has the effect of counteracting the uncertainty in the system. Standard Lyapunov theory is then invoked to show that some calculable compact set, containing the state origin, is uniformly stable, globally strongly attractive, and all solutions of (2.5.1) are globally uniformly bounded.

In order to determine the linear component of the feedback control a further assumption is introduced.

A2.2 : (A,B) is a stabilizable pair.

Thus, under this hypothesis, there exists a $K \in \mathbb{R}^{m\times n}$ such that the state origin of the nominal linear system :

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \ x(t_0)=x_0 \ ,$$

is globally uniformly asymptotically stable with feedback control $u(t): = Kx(t)$. From a well known result (see, for example, Kalman and Bertram [42], corollary 3.2), since $\bar{A} = A + BK$ is asymptotically stable, given a symmetric $Q > 0$, there exists a unique, symmetric solution $P > 0$ of the ***Lyapunov equation***:

$$P\bar{A} + \bar{A}^T P + Q = 0 \ , \quad P,Q \in \mathbb{R}^{n\times n} \ . \tag{2.5.2}$$

P is now used in the construction of the nonlinear component of the feedback control. The proposed class C of feedback controls comprises of functions $\varphi_\varepsilon: \mathbb{R}\times X \to U$ of the form

$$\varphi_\varepsilon(t,x) = Kx + p_\varepsilon(t,x) \ , \tag{2.5.3}$$

where, for any given $\varepsilon > 0$, p_ε is any strongly Carathéodory function such that, for all $(t,x) \in \mathbb{R}\times X$,

$$\| p_\varepsilon(t,x) \| \leq \rho(t,x) \tag{2.5.4}$$

and

$$p_\varepsilon(t,x) = \rho(t,x)\|\eta(t,x)\|^{-1}\eta(t,x), \qquad \text{if } \|\eta(t,x)\| > \varepsilon, \tag{2.5.5}$$

where

$$\eta(t,x) = -\rho(t,x)B^T Px \tag{2.5.6}$$

and the scalar function ρ, strongly Carathéodory, satisfies

$$\rho(t,x) \geq \rho_0(t,x) := (1 - \kappa_0)^{-1}[\ \alpha(t,x) + \kappa_0\| Kx \|]. \tag{2.5.7}$$

Remark: A particular example of a function p_ε, with continuous state dependence but not differentiable (see Leitmann [51]), satisfying the above conditions is

$$p_\varepsilon(t,x) := \begin{cases} \rho_0(t,x)\| \eta(t,x) \|^{-1}\eta(t,x), & \text{if } \| \eta(t,x) \| > \varepsilon \\ \\ \rho_0(t,x)\varepsilon^{-1}\eta(t,x), & \text{if } \| \eta(t,x) \| \leq \varepsilon, \end{cases}$$

with $\eta(t,x) := -\rho_0(t,x)B^T Px$.

An example of a function p_ε, with continuous and differentiable state dependence, is

$$p_\varepsilon(t,x) := \rho_0(t,x)(\|\eta(t,x)\| + \varepsilon)^{-1}\eta(t,x), \quad \text{if } \| \eta(t,x) \| \leq \varepsilon,$$

where $\eta(t,x) := -\rho_0(t,x)B^T Px$. This type of function has been used by Ambrosino, Celentano and Garofalo [2],[3] with reference to the problem of controlling a robot to track a desired path.

To ensure feedback stabilizability, an assumption on the residual uncertainty is now imposed.

A2.3 : $\|g_1(t,x)\| \leq \kappa_1\|x\| + \kappa_2$, for all $(t,x) \in R \times X$,

where $\kappa_1, \kappa_2 > 0$ are known constants.

To prove existence of local solutions, the following proposition (see Corless [19], theorem 8.1) is required:

Proposition 2.5.2: If g: $R \times X \times U \to X$ is a Carathéodory function and h: $R \times X \to U$ is a strongly Carathéodory function, then the function f: $R \times X \to X$, defined by

$$f(t,x) := g(t,x,h(t,x)) \quad \text{for all } (t,x) \in R \times X,$$

is Carathéodory.

Theorem 2.5.1: Let assumptions A2.1–3 hold, with

$$\kappa_1 < \tfrac{1}{2}\sigma_{min}(P^{-1}Q)\{\sigma_{min}(P)\sigma_{min}(P^{-1})\}^{\frac{1}{2}}.$$

If $\varphi_\varepsilon \in C$ then, for every uncertainty realization $g \in G$,

the ellipsoid $E_\varepsilon^Q := \{y \in X : \langle y,Py\rangle \leq (\gamma_\varepsilon^Q)^2\}$,

$\gamma_\varepsilon^Q := (\kappa_2\{\sigma_{max}(P)\}^{\frac{1}{2}} + [4\varepsilon\beta^2 + \kappa_2^2\sigma_{max}(P)]^{\frac{1}{2}})/\beta^2$ and

$\beta^2 := \sigma_{min}(P^{-1}Q) - 2\kappa_1\sigma_{max}(P)$

is a globally uniformly asymptotically stable set for system (2.5.1).

Proof: In view of (2.5.1), (2.5.3) and hypothesis A2.1,

$$\dot{x}(t) = h_\varepsilon(t,x(t)), \tag{2.5.8}$$

where

$$
\begin{aligned}
h_{\varepsilon}(t,x(t)) :&= Ax(t) + B[Kx(t) + p_{\varepsilon}(t,x(t))] + g_1(t,x(t)) \\
&\qquad + Bg_2(t,x(t),Kx(t)+p_{\varepsilon}(t,x(t))) \\
&= \bar{A}x(t) + B[p_{\varepsilon}(t,x(t)) + g_2(t,x(t),Kx(t)+p_{\varepsilon}(t,x(t)))] \\
&\qquad + g_1(t,x(t))
\end{aligned}
$$

As a consequence of hypothesis A2.1 and the definition of the class of feedback controls C, it follows, from proposition 2.5.2, that h_{ε} is a Carathéodory function. Hence, for each $(t_0,x_0) \in \mathbb{R}\times X$, the existence of at least one local solution is guaranteed (see proposition 2.2.1). To demonstrate continuation of solutions, let x: $[t_0,t_1) \to X$ be any maximal solution of (2.5.8), with $x(t_0) = x_0$ and let $\tilde{V}$: $[t_0,t_1) \to [0,\infty)$ be defined by $\tilde{V}(t) := (V \circ x)(t)$, with $V(x) := \langle x,Px\rangle$. Then, since P is symmetric,

$$
\begin{aligned}
\dot{\tilde{V}}(t) &= 2\langle x(t),Ph_{\varepsilon}(t,x(t))\rangle \\
&= 2\{\langle x(t),P\bar{A}x(t)\rangle + \langle x(t),PBp_{\varepsilon}(t,x(t))\rangle + \langle x(t),Pg_1(t,x(t))\rangle \\
&\qquad + \langle x(t),PBg_2(t,x(t),K\ddot{x}(t)+p_{\varepsilon}(t,x(t)))\rangle\} \qquad (2.5.9)
\end{aligned}
$$

Since P satisfies (2.5.2), $2\langle x,P\bar{A}x\rangle = -\langle x,Qx\rangle$ and hypothesis A2.1(ii), together with (2.5.4), implies

$$
\langle x,PBg_2(t,x,Kx+p_{\varepsilon}(t,x))\rangle \le \|B^TPx\| \, [\kappa_0(\|Kx\| + \rho(t,x)) + \alpha(t,x)].
$$

Utilizing (2.5.5) and (2.5.6),

$$
\langle B^TPx,p_{\varepsilon}(t,x)\rangle = -\rho(t,x)\|B^TPx\|, \qquad \text{if } \|\eta(t,x)\| > \varepsilon.
$$

Thus, if $\|\eta(t,x(t))\| > \varepsilon$,

$$
\begin{aligned}
\dot{\tilde{V}}(t) &\le -\langle x(t),Qx(t)\rangle + 2\langle x(t),Pg_1(t,x(t))\rangle \\
&\qquad + 2\|B^TPx(t)\| \, \{\kappa_0\|Kx(t)\| - (1-\kappa_0)\rho(t,x(t)) + \alpha(t,x(t))\} \\
&\le -\langle x(t),Qx(t)\rangle + 2\langle x(t),Pg_1(t,x(t))\rangle, \quad \text{in view of (2.5.7).}
\end{aligned}
$$

If $\|\eta(t,x(t))\| \le \varepsilon$, then, using (2.5.6),

$$\dot{\tilde{V}}(t) \leq -\langle x(t),Qx(t)\rangle + 4\|\eta(t,x(t))\| + 2\langle x(t),Pg_1(t,x(t))\rangle$$

$$\leq -\langle x(t),Qx(t)\rangle + 4\varepsilon + 2\langle x(t),Pg_1(t,x(t))\rangle.$$

Hence, using A2.3, it follows that, for almost all t,

$$\dot{\tilde{V}}(t) \leq -\langle x(t),Qx(t)\rangle + 4\varepsilon + 2\|Px(t)\|(\kappa_1\|x(t)\| + \kappa_2). \tag{2.5.10}$$

From proposition 1.5.7, $\langle x,Qx\rangle \geq \sigma_{\min}(P^{-1}Q)V(x)$,

$\|x\| \leq \{\sigma_{\min}(P)\}^{-\frac{1}{2}}\{V(x)\}^{\frac{1}{2}}$ and $\|Px\| \leq \{\sigma_{\max}(P)\}^{\frac{1}{2}}\{V(x)\}^{\frac{1}{2}}$.

From (2.5.10) it follows that

$$\dot{\tilde{V}}(t) \leq -(\sigma_{\min}(P^{-1}Q) - 2\kappa_1\{\sigma_{\max}(P)/\sigma_{\min}(P)\}^{\frac{1}{2}})\tilde{V}(t)$$
$$+ 2\kappa_2\{\sigma_{\max}(P)\}^{\frac{1}{2}}\{\tilde{V}(t)\}^{\frac{1}{2}} + 4\varepsilon. \tag{2.5.11}$$

Thus, if $\kappa_1 < \frac{1}{2}\sigma_{\min}(P^{-1}Q)\{\sigma_{\max}(P)/\sigma_{\min}(P)\}^{-\frac{1}{2}}$

$\equiv \frac{1}{2}\sigma_{\min}(P^{-1}Q)\{\sigma_{\min}(P)\sigma_{\min}(P^{-1})\}^{\frac{1}{2}}$, from proposition 1.5.8,

then

$$0 < \sigma_{\min}(P^{-1}Q) - 2\kappa_1\{\sigma_{\max}(P)/\sigma_{\min}(P)\}^{\frac{1}{2}} := \beta^2 \text{ , say.} \tag{2.5.12}$$

Hence (2.5.11) may be rewritten as

$$\dot{\tilde{V}}(t) \leq -(\beta\{\tilde{V}(t)\}^{\frac{1}{2}} - \kappa_2\beta^{-1}\{\sigma_{\max}(P)\}^{\frac{1}{2}})^2 + 4\varepsilon + \kappa_2^2\beta^{-2}\sigma_{\max}(P) \tag{2.5.13}$$

from which, and proposition 2.2.2, it may be concluded that x: $[t_0,t_1) \to X$ can be extended into a solution on $[t_0,\infty)$, since finite escape times are precluded by (2.5.13). Moreover, by (2.5.13) and as a direct consequence of lemma 2.3.1, any compact set containing the ellipsoid $E_\varepsilon^Q := \{y \in X : \langle y,Py\rangle \leq (\gamma_\varepsilon^Q)^2\}$, where $\gamma_\varepsilon^Q := (\kappa_2\{\sigma_{\max}(P)\}^{\frac{1}{2}} + [4\varepsilon\beta^2 + \kappa_2^2\sigma_{\max}(P)]^{\frac{1}{2}})/\beta^2$ and β^2 is defined by (2.5.12), is globally uniformly asymptotically stable.

□

Noting that, when $\kappa_2 = 0$, $y \in E_\varepsilon^Q \Rightarrow \|y\|^2 < 4\varepsilon\sigma_{max}(P^{-1})/\beta^2$ it follows for any $r > 0$, there exists $\varepsilon > 0$ and $\delta > 0$ such that $N(E_\varepsilon^Q,\delta) \subset r\bar{B}_X$. Hence the ensuing corollary may be deduced.

Corollary 2.5.1: For any feedback-controlled system modelled by (2.5.1), such that A2.1-3 hold with $\kappa_2 = 0$, given any $r > 0$, there exists a control $\varphi_\varepsilon \in C$ such that $r\bar{B}_X$ is globally uniformly finite-time stable, for arbitrary $g \in G$.

Remark: Q and ε are open to choice and hence can be regarded as design parameters.

Consider the above analysis applied to a linear system (*viz.* the linear regulator) with input uncertainty:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -ax_1(t) - bx_2(t) + u(t) + g(t,u(t)),$$

where $a, b > 0$ are constants and the uncertainty in the input is bounded by

$$\| g(t,u) \| \leq \kappa\|u\| + \alpha(t) , \tag{2.5.14}$$

for $\kappa < 1$ a known constant and α: $\mathbb{R} \to \mathbb{R}$ a known continuous function. Hypothesis A2.1 is satisfied with $\kappa_1 = \kappa_2 = 0$ and $\kappa_0 = \kappa$. In this case, all uncertainty is matched. Since (A,B) is a controllable pair, hypothesis A2.2 holds. Now $\sigma(A) \subset \mathbb{C}^-$ and hence A is asymptotically stable.

Choose $Q = 2I$ and

$$P = (ab)^{-1} \begin{bmatrix} b^2+a(1+a) & b \\ b & 1+a \end{bmatrix} .$$

A class C of continuous stabilizing feedback controls comprise functions of the form :

$$\varphi_\varepsilon(t,x) = \begin{cases} \zeta(t,x)|\eta(t,x)|^{-1}, & |\eta(t,x)|>\varepsilon \\ \\ \zeta(t,x)(|\eta(t,x)|+\varepsilon)^{-1}, & |\eta(t,x)|\leq\varepsilon, \end{cases}$$

where $\zeta(t,x) = (1-\kappa)^{-1}\alpha(t)\eta(t,x)$ and $\eta(t,x) = -(1-\kappa)^{-1}\alpha(t)(bx_1+(1+a)x_2)/(ab)$. By theorem 2.5.1, for arbitrary input uncertainty g satisfying (2.5.14), the ellipsoid $\{ x \in R^2 : \langle x,Px\rangle \leq 2\varepsilon\sigma_{max}(P) \}$ is globally uniformly asymptotically stable.

2.6 Extension to model-following

In this section, robust nonlinear model-following controls are developed for a class of uncertain dynamical systems. The analysis and synthesis procedure follows closely that developed in the previous section and henceforth full identification of the uncertain elements is not required. Other approaches to the model-following problem are discussed in, for example, Balestrino, DeMaria and Zinober [8], Chan [15], Erzberger [23], Landau [44], Shaked [74], Young [90], Zinober [92], and Zinober, El-Ghezawi and Billings [93].

In general, there are three main approaches to the model-following problem for uncertain dynamical systems. ***Variable structure system*** theory has been used to investigate model-following systems by, for example, Young [90] and Zinober [92]. The variable structure feedback control laws are discontinuous in nature, which result in the state "sliding" along a time-varying switching surface. While in the sliding mode, the feedback system is less sensitive to system parameter variations and disturbance inputs. Hence this technique is used for systems where the uncertainty is in the form of external disturbances acting on the system and variations of parameters in the system. In practice, ideal sliding does not occur.

Instead, system motion overshoots the switching surface and is then directed back towards the switching surface. This happens repeatedly so that a so-called "chattering" motion occurs in the sliding mode. Ambrosino, Celentano and Garofalo [2],[3] have shown that the undesirable chatter associated with the sliding mode can be removed using feedback controls which are continuous in the neighbourhood of the switching surface.

An alternative approach is taken by Balestrino, DeMaria and Zinober [8] who show how ***hyperstability*** theory can be used to synthesize discontinuous adaptive control laws for uncertain nonlinear systems. The hyperstability approach is to express the error system in a linear form and to introduce a fictitious output to the system. If appropriate hypotheses, concerning (*i*) the transfer function of the linear system (*ii*) a passivity condition (i.e. an integral inequality constraint), are satisfied then Lyapunov theory can be invoked to ensure global asymptotic stability of the model reference adaptive control system.

A third method, used by Corless, Goodall, Leitmann and Ryan [20], draws on concepts from both variable structure systems theory and Lyapunov based theory and is described in this section. In chapter 4, this work is adapted to discontinuous controls.

Consider uncertain dynamical systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + g(t,x(t),u(t)) , \quad x(t) \in X, \; u(t) \in U, \qquad (2.6.1)$$

where the pair (A,B) define a nominal linear system and the unknown function g belongs to an uncertainty set *G*. An ideal model *M* to be emulated by (2.6.1) is specified *a priori* as the nonlinear system

$$M\colon \; \dot{z}(t)= A^{*}z(t)+B^{*}w(t)+g^{*}(t,z(t),w(t)), \; z(t) \in X, \; w(t) \in \mathbb{R}^{p}, \qquad (2.6.2)$$

where the prescribed triple (A^{*},B^{*},g^{*}) is such that the ensuing feasibility hypothesis holds.

A2.4 : Feasibility of the model :

(*i*) Global existence and uniqueness of model solutions:

For each $(t_0,z_0,w) \in \mathbb{R}\times X\times L^\infty(\mathbb{R};\mathbb{R}^p)$, there exists a unique absolutely continuous function $z(\cdot) = Z(\cdot,t_0,z_0,w)$: $[t_0,\infty) \to X$ satisfying (2.6.2) a.e., with $z(t_0) = z_0$, where Z is the model state transition map.

(*ii*)Model–following conditions (see, for example, Chan [15], Erzberger [23], and Shaked [74]):

(a) $\mathrm{im}(A^*-A) \subset \mathrm{im}(B)$

(b) $\mathrm{im}(B^*) \subset \mathrm{im}(B)$

(c) g^*: $\mathbb{R}\times X\times\mathbb{R}^p \to \mathrm{im}(B)$ is Carathéodory.

The problem to be considered may now be stated as : determine a class C of strongly Carathéodory functions (i.e. feedback controls) φ: $\mathbb{R}\times X\times X\times\mathbb{R}^p \to U$ such that, given any neighbourhood N of the origin in X, there exists $\varphi \in C$ such that, for any $(t_0,z_0,w) \in \mathbb{R}\times X\times L^\infty(\mathbb{R};\mathbb{R}^p)$ and any $g \in G$, the feedback–controlled system

$$S:\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B\varphi(t,x(t),z(t),w(t)) \\ &\qquad + g(t,x(t),\varphi(t,x(t),z(t),w(t))) \\ z(t) &= Z(t,t_0,z_0,w) \end{aligned}\right\} \quad (2.6.3)$$

follows M to within N in the sense of the following definition:

Defn.2.6.1:The system S ***follows*** M ***to within*** N if (*i*) for each $(t_0,x_0) \in \mathbb{R}\times X$, there exists a solution x: $[t_0,t_1) \to X$ of (2.6.3), with $x(t_0)=x_0$, and every such solution can be extended into a solution on $[t_0,\infty)$, (*ii*) every neighbourhood of N is globally uniformly finite–time stable w.r.t. the error $(x(\cdot)-z(\cdot))$ dynamics.

Remark: Clearly, the above problem could also be regarded as that of tracking a prescribed function given by $z(\cdot) = Z(\cdot, t_0, z_0, w)$ and, as such, is closely related to the problem considered by Corless, Leitmann and Ryan [22]. One essential distinction is that, in the present context of model-following, the proposed controls depend on the instantaneous values of the model input $w(t)$ and state $z(t)$ (see Corless, Goodall, Leitmann and Ryan [20]); the model structure is specified *a priori*.

In order to achieve the model-following objective further assumptions are now introduced, *viz*. A2.1-3 are required to hold and all uncertainty in the system is assumed to be matched (i.e. $g_1 \equiv 0$). By A2.2, there exists $K \in R^{m \times n}$ such that $\bar{A} := A + BK$ is asymptotically stable (i.e. with spectrum $\sigma(\bar{A}) \subset \mathbb{C}^-$); hence, given a symmetric $Q > 0$, there exists a unique, symmetric $P > 0$ such that

$$P\bar{A} + \bar{A}^T P + Q = 0. \tag{2.6.4}$$

Recalling that B is of full rank, the linear operator

$$\Pi := (B^T P B)^{-1} B^T P \tag{2.6.5}$$

is well defined, whence, in view of A2.4(*ii*),

$$A^* - A = BK^* , \quad \text{where } K^* := \Pi(A^* - A) \tag{2.6.6}$$

$$B^* = BL^* , \quad \text{where } L^* := \Pi B^* \tag{2.6.7}$$

$$g^* = Bg_2^* , \quad \text{where } g_2^* := \Pi g^* \text{ is Carathéodory.} \tag{2.6.8}$$

Remark: Any linear operator of the form $\Pi = (B^T \Gamma B)^{-1} B^T \Gamma$, with $\Gamma > 0$, would suffice in (2.6.6)-(2.6.8); the particular choice $\Gamma = P$ in (2.6.5) is made for later convenience.

The proposed class C of controls is now comprised of functions φ_ε: $R \times X \times X \times R^p \to U$ of the form

$$\varphi_\varepsilon(t,x,z,w) = \psi(t,x,z,w) + p_\varepsilon(t,x,z,w) \tag{2.6.9}$$

where $\psi(t,x,z,w) := K(x-z) + K^* z + g_2^*(t,z,w) + L^* w$ (2.6.10)

and, with $\varepsilon \in (0,\infty)$, p_ε is any strongly Carathéodory function such that, for all $(t,x,z,w) \in R \times X \times X \times R^p$,

$$\| p_\varepsilon(t,x,z,w) \| \leq \rho^*(t,x,z,w) \tag{2.6.11}$$

and, if $\|\eta(t,x,z,w)\| > \varepsilon$,

$$p_\varepsilon(t,x,z,w) = \rho^*(t,x,z,w)\|\eta(t,x,z,w)\|^{-1}\eta(t,x,z,w), \tag{2.6.12}$$

where $\eta(t,x,z,w) = -\rho^*(t,x,z,w)B^T P(x-z)$ (2.6.13)

and the continuous scalar function ρ^*: $R \times X \times X \times R^p \to R_0^+$ satisfies

$$\rho^*(t,x,z,w) \geq \rho_0^*(t,x,z,w) := (1-\kappa_0)^{-1}[\alpha(t,x)+\kappa_0\|\psi(t,x,z,w)\|]. \tag{2.6.14}$$

Theorem 2.6.1: Suppose A2.1–2, A2.4 hold and $g_1 \equiv 0$. Let $\varphi_\varepsilon \in C$, then, for arbitrary $g \in G$, the feedback-controlled system S follows the model M to within every neighbourhood containing the ellipsoid

$$\{y \in X : \langle y,Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P) \}.$$

Proof: Writing $e(t) = x(t) - z(t)$, then in view of (2.6.2), (2.6.3), (2.6.9) and (2.6.10),

$$\dot{e}(t) = \bar{A}e(t) + B[p_\varepsilon(t,x(t),z(t),w(t)) + g_2(t,x(t),\varphi_\varepsilon(t,x(t),z(t),w(t)))]$$

which, for any given $(t_0,z_0,w) \in \mathbb{R}\times X\times L^\infty(\mathbb{R};\mathbb{R}^p)$, can be expressed as

$$\dot{e}(t) = f_\varepsilon(t,e(t)) \ , \qquad e(t_0) = e_0 := x_0 - z_0 \ , \tag{2.6.15}$$

where

$$f_\varepsilon(t,e) := \bar{A}e + B[p_\varepsilon(t,Z(t,t_0,z_0,w) + e,Z(t,t_0,z_0,w),w(t)) + g(t,Z(t,t_0,z_0,w)+e,\varphi_\varepsilon(t,Z(t,t_0,z_0,w) + e,Z(t,t_0,z_0,w),w(t)) \tag{2.6.16}$$

Hence, the theorem can be established by proving that (*i*) the system (2.6.15)–(2.6.16) has existence and continuation of solutions; (*ii*) the set $\{y \in X : \langle y,Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P)\}$ is globally uniformly finite-time stable. As a consequence of the Carathéodory-type assumptions on the functions of which f_ε is composed, it may be verified that f_ε is Carathéodory. Hence, by similar arguments to those used in establishing theorem 2.5.1, it can be shown that (*i*) for each $(t_0,x_0) \in \mathbb{R}\times X$, the existence of at least one local solution is assured, and (*ii*) $\tilde{V}: t \mapsto (V\circ e)(t)$, with $V(e) := \langle e, Pe\rangle$, satisfies

$$\dot{\tilde{V}}(t) \leq -\sigma_{min}(P^{-1}Q)\tilde{V}(t) + 4\varepsilon \ , \quad \text{a.e.}, \tag{2.6.17}$$

whence, the set

$$\{y \in X: \langle y,Py\rangle \leq 4\varepsilon[\sigma_{min}(P^{-1}Q)]^{-1}\} \equiv \{y \in X: \langle y,Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P)\}$$

(using proposition 1.5.8)) is globally uniformly asymptotically stable for the feedback-controlled system (2.6.15)–(2.6.16).

Finally, from (2.6.17), every trajectory $e: [t_0,\infty) \to X$ must enter after a finite time, independent of t_0, and thereafter remain within every neighbourhood of the ellipsoid

$$\{ y \in X : \langle y, Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P) \}.$$

□

As in corollary 2.5.1, one can immediately deduce :

Corollary 2.6.1: Suppose $g_1 \equiv 0$ and assumptions A2.1–2, A2.4 hold.
Given any neighbourhood N of the origin in X, there exists a control $\varphi_\varepsilon \in C$ such that, for arbitrary $g \in G$, the feedback-controlled system S follows the model M to within N.

Consider now a relaxed version of the model-following controls, i.e. the model-following conditions specified in A2.4(*ii*) are not necessarily satisfied. Defining the projection operator $\Pi_p := I - B\Pi$, where Π is defined as in (2.6.5), which projects on $\ker(B^TP) := \{y \in X: \langle y,Px\rangle = 0 \text{ for all } x \in \text{im}(B)\}$, along im(B), then the following relaxed version of A2.4(*ii*) is introduced.

A2.4$(ii)^*$: Relaxed model-following conditions :

(a) $g^*: \mathbb{R}\times X\times\mathbb{R}^p \to X$ is a Carathéodory function.

(b) For each $(t_0,z_0,w) \in \mathbb{R}\times X\times L^\infty(\mathbb{R};\mathbb{R}^p)$, the function $\zeta^*: [t_0,\infty) \to X$ defined by $\zeta^*(t) := \Pi_p[(A^*-A)z(t) + B^*w(t) + g^*(t,z(t),w(t))]$, with $z(t) = Z(t,t_0,z_0,w)$, is essentially bounded, i.e. there exists $\beta \in [0,\infty)$ such that $\{\langle\zeta^*(t),P\zeta^*(t)\rangle\}^{\frac{1}{2}} \leq \beta$ a.e..

Theorem 2.6.2: Suppose assumptions A2.1–2, A2.4(*i*), A2.4$(ii)^*$ hold and $g_1 \equiv 0$.
If $\varphi_\varepsilon \in C$ then, for arbitrary $g \in G$, the feedback-controlled system S follows the model M to within every neighbourhood containing the ellipsoid

$$E^\beta_\varepsilon := \{y \in X: \langle y,Py\rangle \leq (\gamma^\beta_\varepsilon)^2\},$$

where $\gamma^\beta_\varepsilon := \beta\sigma_{max}(Q^{-1}P) + [\,\{\beta\sigma_{max}(Q^{-1}P)\}^2 + 4\varepsilon\sigma_{max}(Q^{-1}P)]^{\frac{1}{2}}$.

Proof:Let e: $[t_0,t_1) \to X$ be any local solution of (2.6.15)-(2.6.16) and let $\tilde{V}: [t_0,t_1) \to [0,\infty)$ be given by $\tilde{V}(t) := (V \circ e)(t)$ with $V(e) := \langle e, Pe \rangle$. By a straightforward but lengthy calculation (analogous to that in the proof of theorem 2.5.1), using (2.6.4), (2.6.9)-(2.6.16),

$$\dot{\tilde{V}}(t) \leq -\sigma_{min}(P^{-1}Q)\tilde{V}(t) + 4\varepsilon + 2\langle e(t), P\zeta^*(t)\rangle,$$

whence, in view of A2.3$(ii)^*$,

$$\dot{\tilde{V}}(t) \leq -\sigma_{min}(P^{-1}Q)\tilde{V}(t) + 2\beta\{\tilde{V}(t)\}^{\frac{1}{2}} + 4\varepsilon.$$

With similar analysis to that in theorem 2.6.1, the required result can be concluded.

□

Noting that, given any $\delta > 0$, there exists $\varepsilon > 0$ such that $E^\beta_\varepsilon \subset N(E^\beta_0,\delta)$, where

$$E^\beta_0 = \{y \in X: V(y) \leq [2\beta\sigma_{max}(Q^{-1}P)]^2\} \quad ,$$

the following corollary may be stated (see corollary 2.5.1).

Corollary 2.6.2: Let assumptions A2.1-2, A2.4(i), A2.4$(ii)^*$ hold. For any $\delta > 0$, there exists $\varepsilon > 0$ such that, under control $\varphi_\varepsilon \in C$ and arbitrary $g \in G$, the feedback-controlled system S follows M to within the set $N(E^\beta_0,\delta)$.

Remark: If A2.3(ii) holds, then $\zeta^*(t) = 0$ a.e. and $\beta = 0$. Hence, theorem 2.6.1 (corollary 2.6.1) is a consequence of theorem 2.6.2 (see corollary 2.6.2).

2.7 Extension to the tracking problem

The analysis of the previous section can easily be adapted for the problem of tracking a prescribed function $z(\cdot)$. Here a class C of strong Carathéodory feedback control functions φ: $\mathbb{R}\times X\times X \to U$ is determined such that, given any neighbourhood N of the origin in X, there exists $\varphi \in C$ such that, for any $(t_0,x_0) \in \mathbb{R}\times X$ and $g \in G$, the feedback-controlled system

$$\dot{x}(t) = Ax(t) + B\varphi(t,x(t),z(t)) + g(t,x(t),\varphi(t,x(t),z(t))) \qquad (2.7.1)$$

tracks $z(\cdot)$ in the sense of the following definition :

Defn. 2.7.1: The system (2.7.1) ***tracks*** an absolutely continuous function $z(\cdot)$ ***to within*** N if (*i*) for each $(t_0,x_0) \in \mathbb{R}\times X$, there exists a solution x: $[t_0,t_1) \to X$ of (2.7.1), with $x(t_0)=x_0$, and every such solution can be continued indefinitely (*ii*) every neighbourhood of N is globally uniformly finite-time stable w.r.t. the error, $x(\cdot) - z(\cdot)$, dynamics.

As in the case of the model-following problem, it is assumed that all uncertainty in the system is matched and assumptions A2.1-2 hold. To ensure feasibility of tracking $z(\cdot)$, the following hypothesis is required to hold:

A2.5: Feasibility of the motion to track z:

There exists a function $\theta \in L^\infty(\mathbb{R};\mathbb{R}^m)$ such that

$$\dot{z}(t) = Az(t) + B\theta(t) \quad \text{a.e..}$$

The proposed class C of controls are functions φ_ε: $\mathbb{R}\times X \to U$ of the form

$$\varphi_\varepsilon(t,x) = \psi(t,x) + p_\varepsilon(t,x) \;, \qquad (2.7.2)$$

$$\text{where} \quad \psi(t,x) := K(x-z(t)) + \theta(t), \qquad (2.7.3)$$

$\theta(\cdot)$ is any function satisfying A2.5, and, for any $\varepsilon > 0$, p_ε is any strongly Carathéodory function such that, for all $(t,x) \in \mathbb{R}\times X$,

$$\|p_\varepsilon(t,x)\| \leq \rho(t,x) \tag{2.7.4}$$

and, if $\|\eta(t,x)\| > \varepsilon$,

$$p_\varepsilon(t,x) = \rho(t,x)\|\eta(t,x)\|^{-1}\eta(t,x) \tag{2.7.5}$$

where $\eta(t,x) = -B^T P(x-z(t))\rho(t,x)$ (2.7.6)

and $\rho(t,x)$ satisfies

$$\rho(t,x) \geq \rho_0(t,x) := (1-\kappa_0)^{-1}[\alpha(t,x) + \kappa_0\|\psi(t,x)\|] \ . \tag{2.7.7}$$

Using a similar argument to that used in the proof of theorem 2.6.1, tracking of $z(\cdot)$ can be assured as shown in the following results.

<u>Theorem 2.7.1</u>: Suppose $g_1 \equiv 0$ and assumptions A2.1–2, A2.5 hold for system (2.7.1) and an absolutely continuous function $z: \mathbb{R} \to X$. If $\varphi_\varepsilon \in C$ then, for arbitrary $g \in G$, the feedback–controlled system given by (2.7.1) and (2.7.2)–(2.7.7) tracks $z(\cdot)$ to within any neighbourhood of the ellipsoid $\{y \in X: \langle y,Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P)\}$.

The parameter ε can be adjusted so that every possible motion of the uncertain feedback system , (2.7.1) and (2.7.2)–(2.7.7), enters an arbitrary small prescribed neighbourhood of the zero state in finite time and remains in it thereafter.

<u>Corollary 2.7.1</u>: Under the conditions stated in theorem 2.7.1, given any neighbourhood N of the origin in X, there exists a control $\varphi_\varepsilon \in C$ such that, for arbitrary $g \in G$, the feedback–controlled system tracks $z(\cdot)$ to within N.

The feasibility of tracking assumption can be relaxed (see Corless, Leitmann and Ryan [22]).

A2.5*:Relaxed tracking conditions:

For each $(t_0,x_0) \in \mathbb{R}\times X$, the function ζ: $[t_0,\infty) \to X$, defined (a.e.) by $\zeta(t) := \Pi_P[\dot{z}(t) - Az(t)]$ is essentially bounded.

Theorem 2.7.2:Suppose assumptions A2.1–2, A2.5* hold and let θ: $\mathbb{R} \to \mathbb{R}^m$ satisfy a.e. $\theta(t) = \Pi[\dot{z}(t) - Az(t)]$. If $\varphi_\varepsilon \in C$ then, for arbitrary $g \in G$ (with $g_1 \equiv 0$), the feedback-controlled system (2.7.1)–(2.7.7) tracks an absolutely continuous function z: $\mathbb{R} \to X$ to within any neighbourhood of the ellipsoid E^{β}_{ε} (defined in theorem 2.6.2), where the constant β is such that

$$\{\langle\zeta(t),P\zeta(t)\rangle\}^{\frac{1}{2}} < \beta \quad \text{a.e..}$$

Proof: See theorem 2.6.2..

□

With reference to corollary 2.6.2, one can deduce

Corollary 2.7.2:Let assumptions A2.1–2, A2.5* hold. For any $\delta > 0$, there exists $\varepsilon > 0$ such that, under control $\varphi_\varepsilon \in C$ and arbitrary $g \in G$ ($g_1 \equiv 0$), the feedback-controlled system (2.7.1)–(2.7.7) tracks $z(\cdot)$ to within the set $N(E^{\beta}_{0},\delta)$.

2.8 Lorenz model-following example

As a particular example to illustrate the model-following theory developed in §2.6, consider (2.6.1) with

$$A = \begin{bmatrix} \mu & -\mu & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \quad \mu \neq 0, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.8.1}$$

It is required to follow a model M defined by the Lorenz equations, *viz.* of the form (2.6.2) with

$$A^* = \begin{bmatrix} \mu & -\mu & 0 \\ r & -1 & 0 \\ 0 & 0 & -s \end{bmatrix}, \quad B^* = 0$$

$$f^*(t,z,w) = \hat{f}(z) := \begin{bmatrix} 0 \\ -z_1 z_3 \\ z_1 z_2 \end{bmatrix}$$

Remark: This model is adopted here in order to illustrate the robustness of the proposed feedback controls. The Lorenz equations, while of deceptively simple structure, are well known to exhibit immense richness in dynamic behaviour ("chaos", "strange attractors", etc., see for example Sparrow [77] and the bibliography therein) and thus provide a demanding test case for the proposed class C of controls.

For the Lorenz model, A2.3(i) holds (see Sparrow [77]); in view of (2.8.1), the model-following conditions A2.3(ii) also hold, and

$$K^* = \begin{bmatrix} r & -(1+\lambda_1) & 0 \\ & & \\ 0 & 0 & -(s+\lambda_2) \end{bmatrix}, \quad L^* = 0$$

$$g^*(t,z,w) = \hat{g}(z) := \begin{bmatrix} -z_1 z_3 \\ \\ z_1 z_2 \end{bmatrix}.$$

Now, clearly the pair (A,B) in (2.8.1) is controllable, and hence *a fortiori* stabilizable, so that A2.2 holds. Finally, A2.1 is assumed to hold (with $g_1 \equiv 0$) and for simplicity $\kappa_0 = \frac{1}{4}$, $\alpha(t,x) = 10 \quad \forall(t,x)$. Adopting the parameter values $\mu = -10$, $\lambda_1 = -5$, $\lambda_2 = -1$, $s = 8/3$ then A has spectrum

$$\sigma(A) = \{-10,-5,-1\} \subset \mathbb{C}^-$$

and hence the choice K = O is admissible. Selecting Q = I, then, by (2.6.4),

$$P = \frac{1}{60}\begin{bmatrix} 3 & 2 & 0 \\ 2 & 10 & 0 \\ 0 & 0 & 30 \end{bmatrix}$$

and, in view of (2.6.10) and (2.6.14),

$$\psi(t,x,z,w) = \hat{\psi}(z) := \begin{bmatrix} rz_1 + 4z_2 - z_1 z_3 \\ \\ -(5/3)z_3 + z_1 z_2 \end{bmatrix}$$

$$\rho_0(t,x,z,w) = \hat{\rho}_0(z) := (4/3)[\ 10 + \tfrac{1}{4}\|\hat{\psi}(z)\|\ ].$$

Finally, adopting a function p_ε of the form (2.6.12)–(2.6.13), the class C of controls is comprised of functions of the form

$$\varphi_\varepsilon(t,x,z,w) = \hat{\varphi}_\varepsilon(x,z) := \hat{\psi}(z) + \hat{p}_\varepsilon(x,z),$$

$$\text{where}\quad \hat{p}_\varepsilon(x,z) := \begin{cases} \hat{\rho}_0(z)\|\hat{\eta}(x,z)\|^{-1}\hat{\eta}(x,z), & \|\hat{\eta}(x,z)\| > \varepsilon \\ \\ \hat{\rho}_0(z)\varepsilon^{-1}\hat{\eta}(x,z), & \|\hat{\eta}(x,z)\| \le \varepsilon \end{cases}$$

with $\hat{\eta}(x,z) := -\hat{\rho}_0(z)B^TP[x-z]$.

The effectiveness of this control design is clearly illustrated in the figures 1, 2 and 3 (shown below) which depict, componentwise, the computed evolution of system $x(\cdot)$ and model $z(\cdot)$ trajectories, together with the error norm function $\|e(\cdot)\| = \|x(\cdot) - z(\cdot)\|$, for the model parameter values $r = 10$, 30, and 50, respectively. In each case, the model and system initial data is given by $z^0 = [10 \quad 5 \quad 40]^T$, $x(t_0) = -z^0$, the controller parameter value is $\varepsilon = 0.01$ and an admissible uncertainty realization $g \in G$ is given by

$$g(t,x,u) = \frac{1}{4}\begin{bmatrix} 0 \\ u_1 \\ u_2 \end{bmatrix} + \frac{10}{\sqrt{2}}\begin{bmatrix} 0 \\ \sin(x_1/100) \\ \cos(x_3/100) \end{bmatrix} .$$

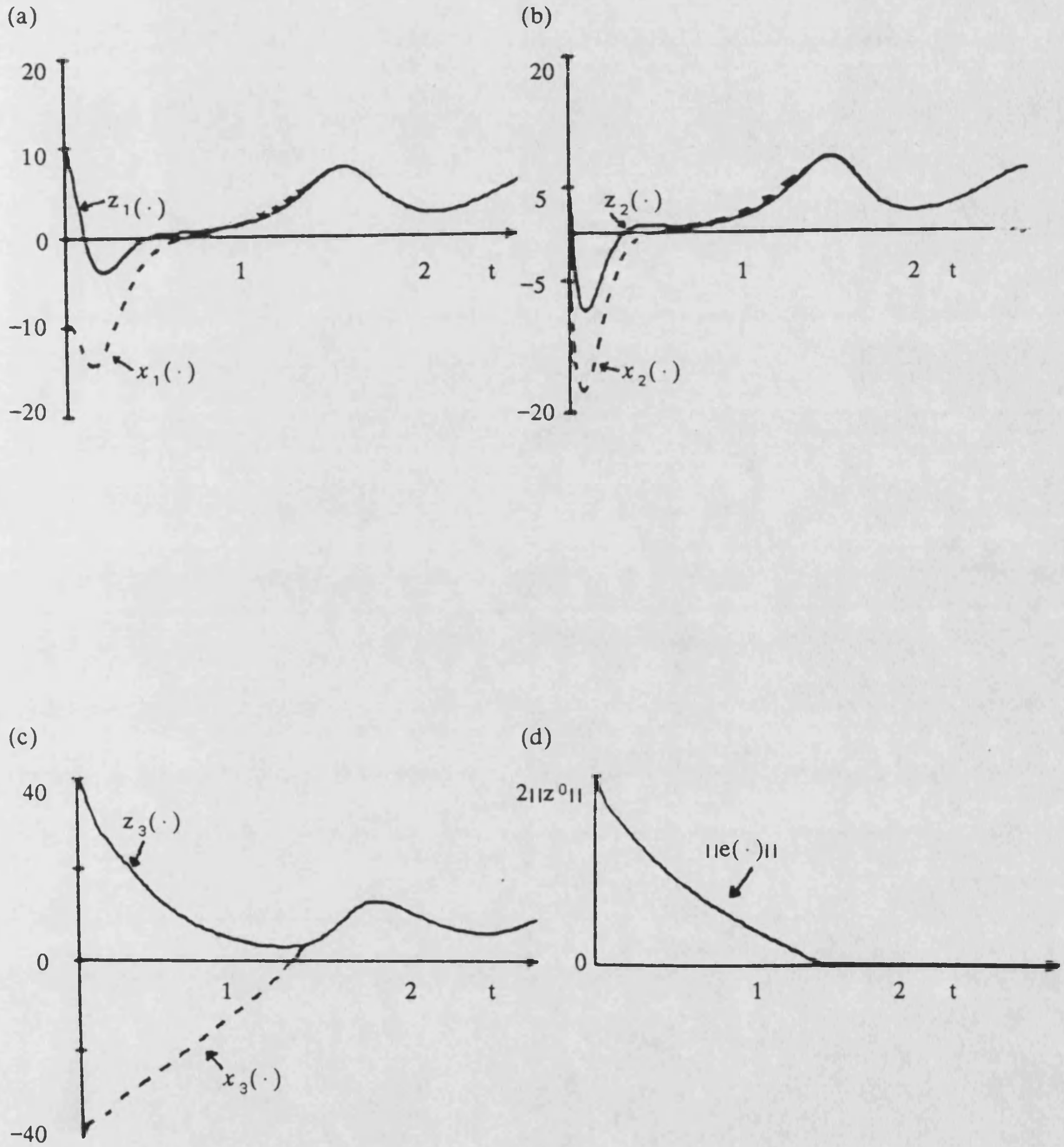

Fig. 1. Model and system trajectories for parameter value r = 10.

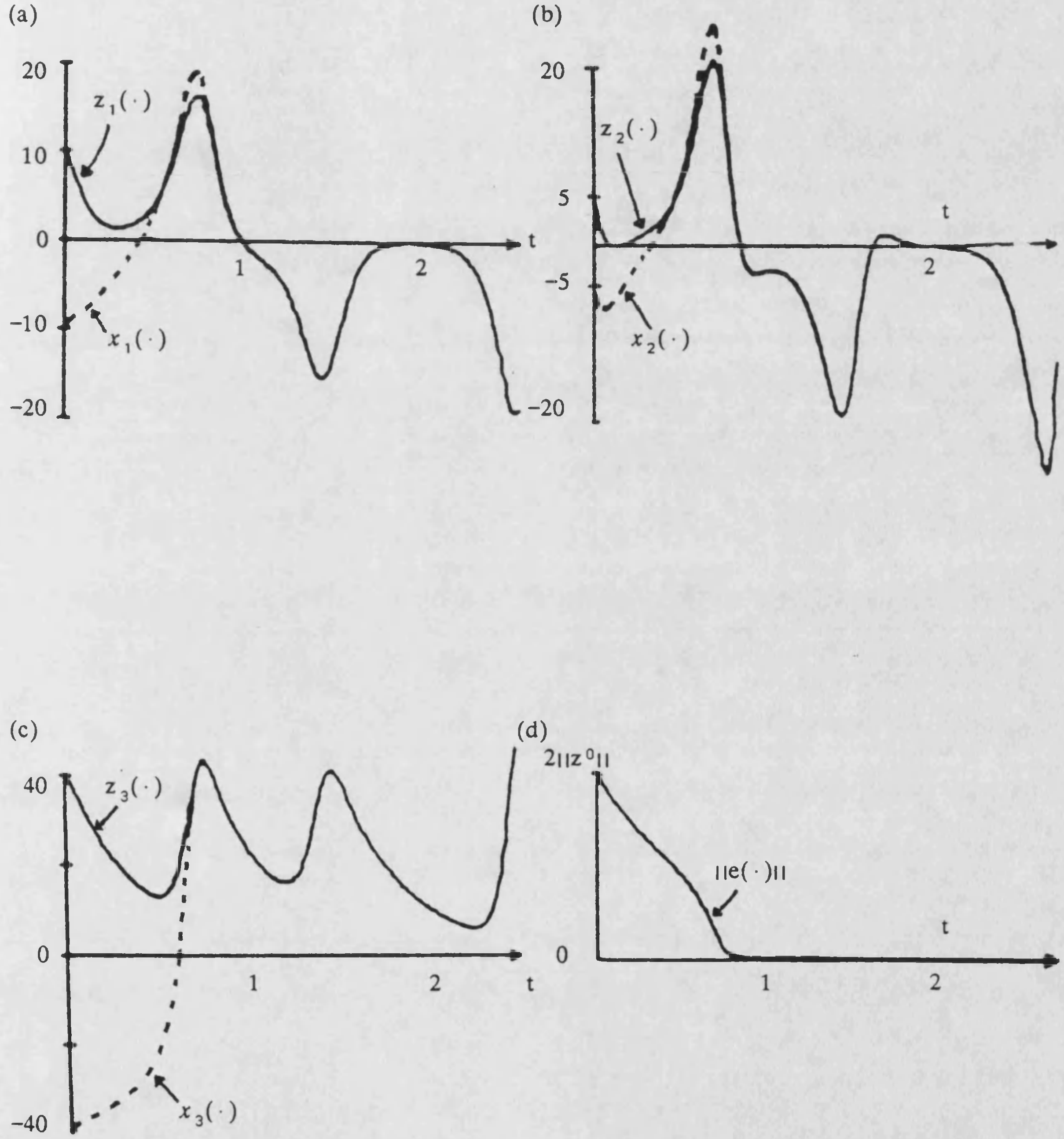

Fig. 2. Model and system trajectories for parameter value r = 30.

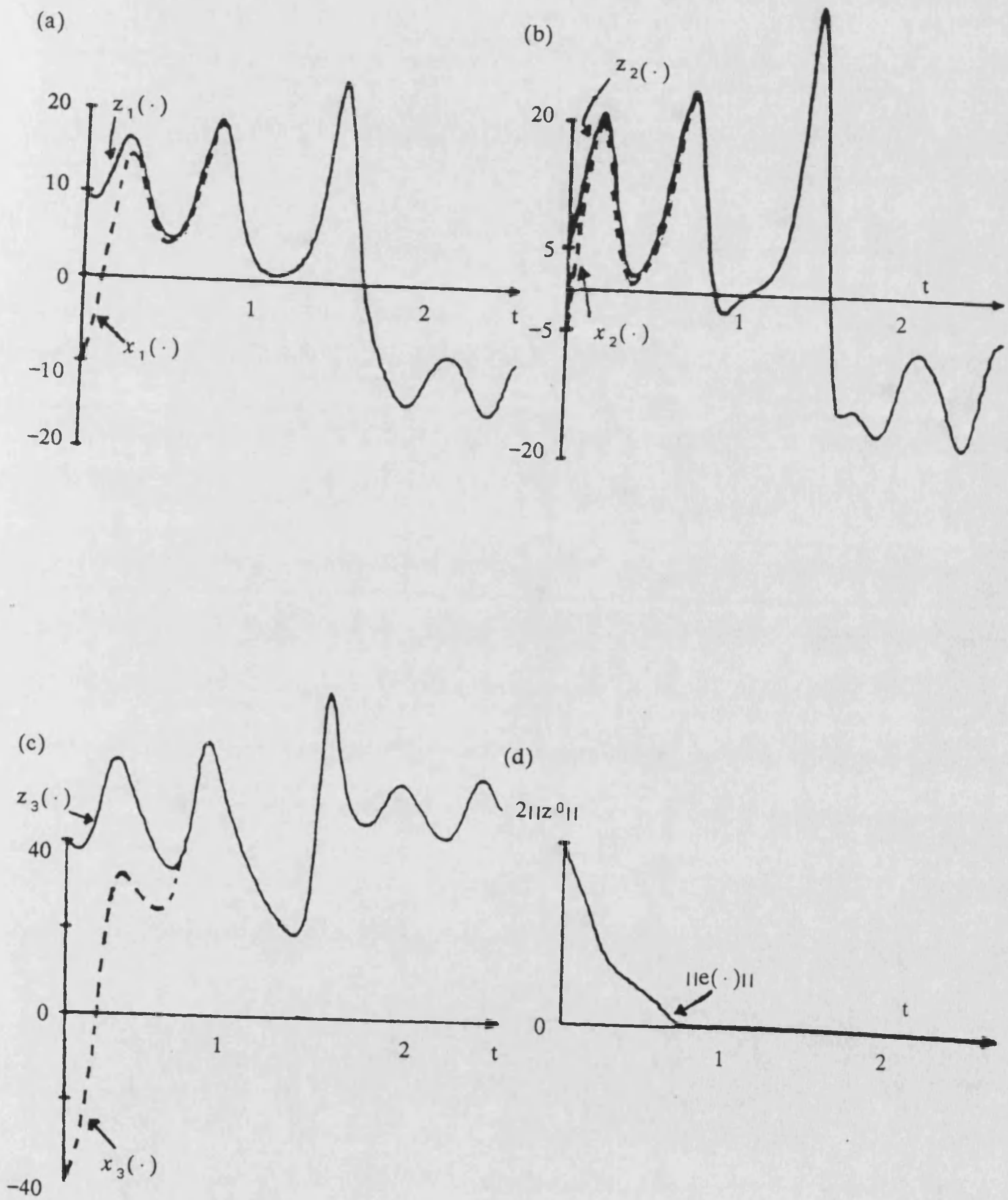

Fig. 3. Model and system trajectories for parameter value r = 50.

3.DETERMINISTIC CONTROL OF UNCERTAIN SYSTEMS MODELLED BY DIFFERENTIAL INCLUSIONS

3.1 Introduction

In the previous chapter, nonlinear continuous controls were designed for a class of uncertain dynamical systems, modelled by nonlinear differential equations of the form

$$\dot{x}(t) = h(t,x(t),u(t)) \ , \quad x(t_0)=x_0, \qquad x(t) \in X, \ u(t) \in U. \qquad (3.1.1)$$

In general, dynamical systems cannot always be precisely defined since some approximation, imprecision or uncertainty may have been introduced during the modelling procedure. Thus, strictly speaking, $h(t,x,u) \in H(t,x,u)$, where $(t,x,u) \mapsto H(t,x,u)$ is a ***multifunction*** (*viz.* a multi-valued function or set-valued map) which is identified with the true controlled vector field. Thus, (3.1.1) may be replaced by the controlled ***differential inclusion***

$$\dot{x}(t) \in H(t,x(t),u(t)) \ , \quad x(t_0)=x_0, \quad x(t) \in X, \ u(t) \in U. \qquad (3.1.2)$$

Consider , for example, the feedback-controlled system described in §2.4, *viz.*

$$\dot{x}(t)= Ax(t) + g_1(t,x(t)) + B[u(t)+g_2(t,x(t),u(t))], \quad x(t)\in X, \ u(t)\in U,$$

where $g_1(t,x)$ and $g_2(t,x,u)$ satisfy

$$(i) \ \|g_1(t,x)\| \leq \kappa_1\|x\| + \kappa_2 \ , \qquad \forall(t,x) \in R\times X$$

$$(ii) \|g_2(t,x,u)\| \leq \kappa_0\|u\| + \alpha(t,x) \ , \qquad \forall(t,x,u) \in R\times X\times U,$$

$\alpha(t,x)$ is a known Carathéodory function and $\kappa_0, \ \kappa_1, \ \kappa_2 > 0$ are known constants, with $\kappa_0 < 1$. This system can be reformulated as the differential inclusion system :

$$\dot{x}(t) \in Ax(t) + \{\kappa_1\|x(t)\| + \kappa_2\}\bar{B}_X + B[u(t) + \{\kappa_0\|u(t)\| + \alpha(t,x(t))\}\bar{B}_U].$$

Thus, in this chapter, the problem of stabilization of a class of uncertain dynamical systems, modelled by a differential inclusion is considered.

There are other instances where a differential inclusion formulation is more suitable. For instance, consider discontinuous control which is a natural candidate in many problems of stabilization and optimization, where the associated differential equation (3.1.1), modelling the feedback controlled system, fails to satisfy the requisite hypotheses of classical theory. In this case, $h(t,x,F(t,x))$ can be embedded in a multifunction

$$(t,x) \mapsto H_F(t,x) := \{\ h(t,x,u):\ u \in F\ \},$$

with $H_F(t,x)$ sufficiently regular, such that the analytical difficulties relating to the trajectories of (3.1.1), alluded to above, are overcome (see chapter 4). The solutions to (3.1.1) are then solutions to the differential inclusion :

$$\dot{x}(t) \in H_F(t,x(t)), \qquad x(t_0) = x_0 \ . \tag{3.1.3}$$

Differential inclusions also arise naturally in many other ways. For example, consider the implicit differential equation

$$f(t,x,\dot{x}) = 0\ , \qquad x(t_0)=x_0.$$

This can be rewritten as the differential inclusion

$$\dot{x}(t) \in F(t,x(t))\ , \quad x(t_0)=x_0,$$

where $F(t,x) := \{\ v:\ f(t,x,v) = 0\ \}$.

As another example, the problem of solving the differential inequality

$$\|\ \dot{x}(t) - g(t,x(t))\ \| \leq f(t,x(t)),$$

where $g\colon R \times X \to X$ and $f\colon R \times X \to R_0^+$, is equivalent to solving the differential inclusion

$$\dot{x}(t) \in F(t,x(t)) \ ,$$

where $F(t,x) := g(t,x) + f(t,x)\bar{B}_X$.

Stabilization of controlled differential inclusions, using Lyapunov techniques, have been discussed by Aubin and Cellina [7], Gutman [31], Gutman and Palmor [33], and Leitmann [51]. With reference to a differential inclusion system of the form (3.1.3), stability concepts and criteria are given in §3.2, whilst the precise structure of the differential inclusion model of the uncertain system is described in §3.3. The problem of feedback stabilization is considered in §3.4, where a class of ***(generalized)*** feedback controls is introduced and each feedback control is robust w.r.t. arbitrary uncertainty realizations present in the system. A generalized feedback is a prescribed multifunction which encompasses a set of continuous feedback controls. The concept of a generalized feedback is very useful, since it is defined in such a way that it also encompasses discontinuous controls. The stability of differential inclusion control systems using discontinuous controls is examined in chapter 4. In §3.5 the model-following example (discussed in §2.6) is recast in terms of differential inclusions and, in §3.6, the tracking problem, introduced in §2.7, is reformulated as a tracking problem for a differential inclusion system. Finally, an application from robotics is chosen to illustrate the theory described in this chapter.

3.2 Stability concepts and criteria with respect to differential inclusions

Consider the differential inclusion

$$\dot{x}(t) \in H_F(t,x(t)) \ , \ x(t_0)=x_0, \quad x(t) \in X. \qquad (3.2.1)$$

Definitions of : (*i*) a local (or global) solution, (*ii*) existence of local solutions, (*iii*) indefinite continuation of solutions, are essentially the same as those defined for systems modelled by differential equations (see §2.2). For example, a local solution of the differential inclusion (3.2.1) is any absolutely continuous function $x(\cdot)$: $[t_0,t_1) \to X$ satisfying (3.2.1) a.e. on $[t_0,t_1)$.

In order to investigate existence of (local or global) solutions to a differential inclusion sufficient conditions are required. For the differential equation (3.1.1) it was assumed that h satisfied regularity requirements (in the Carathéodory sense) to ensure existence (see chapter 2). In the case of the differential inclusion (3.2.1), conditions imposed on the multifunction H_F are chosen to be of the form:

(a) H_F is upper semicontinuous

(b) the values of H_F are nonempty, compact and convex.

These conditions on H_F are assumed to hold for the remaining part of this section.

The following proposition (essentially theorem 3 of chapter 2 in Aubin and Cellina [7]) asserts that the above conditions (a) and (b) are sufficient for existence of local solutions.

<u>Proposition 3.2.1</u>:For each (t_0,x_0), there exists a local solution x: $[t_0,t_1) \to X$ of (3.2.1) satisfying $x(t_0)=x_0$.

The next proposition is required for the proof of the analogue of proposition 2.2.2, i.e. sufficient conditions are given for the differential inclusion (3.2.1) to have indefinite continuation of solutions.

<u>Proposition 3.2.2</u>:Every local solution of (3.2.1) can be extended into a maximal solution.

Proof:Let x: $[t_0,\omega) \to X$ be a solution of (3.2.1) for some $t_0 < \omega < \infty$ and the set Υ be defined by

$\Upsilon := \{(\rho,y)\colon \omega < \rho < \infty;\ y:[t_0,\rho) \to X$ solves (3.2.1), $y(t)=x(t)\ \forall\ t \in [t_0,\omega)\}$.

The set Υ is nonempty and a partial ordering of Υ can be defined, through the relation $\leqslant$, by $(\rho_1,y_1) \leqslant (\rho_2,y_2) \iff \rho_1 < \rho_2$ and $y_2(t) = y_1(t)$ $\forall\ t \in [t_0,\rho_1)$. Suppose Φ is a totally ordered subset of Υ, then an upper bound $\upsilon \in \Upsilon$ can be defined for the set Φ: let $\rho^* = \sup\{\rho\colon (\rho,y) \in \Upsilon\}$ and let y^*: $[t_0,\rho^*) \to X$ be defined by the property that, for every $(\rho,y) \in \Upsilon$, $y^*(t) = y(t)$ for all $t \in [t_0,\rho)$. Then $\upsilon = (\rho^*,y^*) \in \Upsilon$ and is an upper bound for Φ. Hence, by Zorn's lemma (see proposition 1.5.10), Υ contains at least one maximal element.

□

Remark:The proof of the above proposition (for the non-autonomous system (3.2.1)) is the same as that for the autonomous case which is given in a paper by Ryan [64].

In order to show that a maximal solution can be extended into a global solution the following proposition may be invoked (this being the analogue of proposition 2.2.2).

Proposition 3.2.3:Let x: $[t_0,t_1) \to X$ be a maximal solution of (3.2.1). If $x(\cdot)$ is bounded, then $t_1=\infty$.

Proof:(See Ryan [64])

By propositions 3.2.1 and 3.2.2, there exists a local solution $x(\cdot)$ of (3.2.1) which can be extended into a maximal solution defined on $[t_0,t_1)$, say. Suppose t_1 is finite. By the boundedness of x, there exists a compact

$K \subset X$ such that $x(t) \in K$, for all $t \in [t_0,t_1)$. By proposition 1.5.5, $\bigcup_{(t,x)\in[t_0,t_1]\times K} H_F(t,x)$ is compact. This implies that there exists $k > 0$ such that $\|\dot{x}(t)\| \leq k$ for almost all $t \in [t_0,t_1)$. Therefore x is uniformly continuous on $[t_0,t_1)$ and, hence, can be extended into a bounded absolutely continuous function on the closed interval $[t_0,t_1]$. By proposition 3.2.1, this function extends into a solution of (3.2.1) on the interval $[t_0,t_2)$, with $t_2>t_1$. This contradicts the maximality of x and therefore $t_1=\infty$.

□

Boundedness and stability of solutions for differential equations are defined in §2.2. These can easily be modified so that boundedness and stability of solutions for differential inclusions can be analogously defined. Similarly, stability concepts relating to a nonempty closed set can be defined for differential inclusion systems in precisely the same form as for differential equations (see §2.2).

Suppose H_F satisfies conditions sufficient for the existence of a local solution which can be continued indefinitely. Stability properties for a compact set M are now obtained for the differential inclusion system (3.2.1) using a nonsmooth Lyapunov candidate $(t,x) \mapsto V(t,x)$, where V is assumed to be locally Lipschitz on $\mathbb{R}\times X$. But first, the analogue of proposition 2.3.1 is required.

<u>Proposition 3.2.4</u>: Suppose $V: \mathbb{R}\times X \to \mathbb{R}$ is locally Lipschitz, then along all maximal solutions of (3.2.1)

$$DV(t,x(t)) = D_+V(t,x(t);\ \dot{x}(t))$$

for almost all t.

<u>Proof</u>: By propositions 3.2.1 and 3.2.2, there exists a maximal solution $x(\cdot)$ of (3.2.1). The remaining part of the proof is analogous to that of proposition 2.3.1.

□

Lemma 3.2.1: Let $M \subset X$ be compact and nonempty. Suppose there exists a locally Lipschitz function $V: \mathbb{R}\times X \to \mathbb{R}$ such that, for all $(t,x) \in \mathbb{R}\times X$, V satisfies :

(a) $V(t,x) = 0$ for all $(t,x) \in \mathbb{R}\times\partial M$

(b) there exists strictly increasing functions $W_1, W_2: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, with $W_1, W_2 \in C(\mathbb{R}_0^+)$, such that $W_1(0) = W_2(0) = 0$ and

$W_2(d(x,M)) \geq V(t,x) \geq W_1(d(x,M))$,

for all $(t,x) \in \mathbb{R}\times(X\backslash M)$

(c) there exists either (i) a strictly increasing continuous function $W_3: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, or (ii) a strictly increasing, upper semicontinuous function $W_3: \mathbb{R}_0^+ \to \mathbb{R}_0^+$, bounded above,

such that,

$$D_+V(t,x;h) + W_3(d(x,M)) \leq 0 \qquad (3.2.2)$$

for all $h \in H_F(t,x)$ and $(t,x) \in \mathbb{R}\times(X\backslash M)$,

(d) $W_1(d(x,M)) \to \infty$ as $d(x,M) \to \infty$,

then the set M is a globally uniformly asymptotically stable set for the system (3.2.1).

Proof: Invoking propositions 3.2.1 and 3.2.2, there exists a local solution $x(\cdot)$ of (3.2.1) which can be extended to a solution on a maximal interval of existence, say $[t_0,t_1)$. Using propositions 3.2.3 and 3.2.4, it is now shown that (3.2.1) has indefinite continuation of solutions. Consider $(t_0,x_0) \in \mathbb{R}\times(X\backslash M)$ and let $[\alpha_0,\alpha_1] \subset [t_0,t_1)$ be arbitrary. As in the proof of lemma 2.3.1, $t \mapsto V(t,x(t))$ is absolutely continuous on $[\alpha_0,\alpha_1]$. Hence, from proposition 3.2.4 and inequality (3.2.2), one can conclude that,

$$V(t,x(t)) = V(\alpha_0,x(\alpha_0)) + \int_{\alpha_0}^{t} DV(t,x(t))ds \leq V(\alpha_0,x(\alpha_0))$$

for all $t \in [\alpha_0,\alpha_1]$. Therefore, since $[\alpha_0,\alpha_1]$ is arbitrary, $V \circ x$ is absolutely continuous and non-increasing on $[t_0,t_1)$. Analogous to the proof of lemma 2.3.1, it can be shown that x is bounded on $[t_0,t_1)$ and thus, by proposition 3.2.3, the differential inclusion (3.2.1) has indefinite continuation of solutions. Uniform stability and weak attractiveness of the set M, as well as global uniform boundedness of solutions (with trajectory in X\M), is proved using the standard arguments illustrated in lemma 2.3.1. Finally, one must show that M is globally uniformly weakly attractive. Using propositions 2.3.2 and 3.2.4 and inequality (3.2.2), one can conclude that, along solutions to (3.2.1),

$$V(t,x(t)) \leq V(t_0,x_0) - \int_{t_0}^{t} W_3(d(x(s),M))\, ds$$

under hypothesis (c) of this lemma. The remaining part of the proof is then analogous to the corresponding part in the proof of lemma 2.3.1.

□

3.3 Differential inclusion system model

As noted in §3.1, if (3.1.1) corresponds to a mathematical model of a "real" process then it is almost certain that some approximation, imprecision or uncertainty will have been introduced during the modelling procedure. The function h can only be regarded as a "reasonable" representation of the controlled vector field. If uncertainty is an intrinsic feature, h will not be known precisely and therefore it is natural to replace (3.1.1) by a controlled differential inclusion of the form :

$$\dot{x}(t) \in H(t,x(t),u(t)) \;, \quad x(t_0)=x_0, \qquad x(t) \in X, \; u(t) \in U \qquad (3.3.1)$$

where $(t,x,u) \mapsto H(t,x,u)$: $\mathbb{R}\times X\times U \rightrightarrows X$ is a known multifunction with nonempty values, which is the set of all possible velocities $\dot{x}(t)$ of the uncertain system at time t. To ensure existence of solutions, conditions of upper semicontinuity, and convexity and compactness of its values, are imposed on H. In order to ensure feedback stabilizability certain structural properties and bounds relating to the uncertainty, described below, are assumed to be known. Suppose that

$$H(t,x,u) = Ax + Bu + G(t,x,u) \ , \qquad \forall (t,x,u), \tag{3.3.2}$$

where the known matrix pair (A,B) defines a ***nominal*** linear system and G is a known multifunction. This defines a class of systems which can be regarded as nonlinear perturbations of the nominal linear system. The following assumptions are now imposed:

A3.1: There exist nonempty multifunctions G_1: $\mathbb{R}\times X \rightrightarrows \ker(B^T)$ and G_2: $\mathbb{R}\times X \rightrightarrows U$ and a real constant $\kappa_0 < 1$ such that

(*i*) $\Pi_{\ker(B^T)} G(t,x,u) = G_1(t,x) \qquad \forall\ (t,x,u)$;

(*ii*) $\Pi_{\mathrm{im}(B)} G(t,x,u) = B[G_2(t,x) + \kappa_0 G_3(u)] \qquad \forall\ (t,x,u)$,

where G_3: $u \mapsto \|u\| \bar{B}_U$;

(*iii*) G_1 and G_2 are upper semicontinuous with convex and compact values.

Remarks: (*i*) In the terminology of Barmish, Corless and Leitmann [9], Barmish and Leitmann [10], the multifunctions G_2 and $\kappa_0 G_3$ may be interpreted as modelling the ***matched*** uncertainty in the system, while G_1 models ***residual*** uncertainty.

(*ii*) A simple example of a class of matched uncertain systems which can be embedded into the more general class considered here is typified by the following :

$$\dot{x}(t) = Ax(t) + B[u(t) + g(t,x(t),u(t))],$$

where g is an unknown function with

$$\|g(t,x,u)\| \leq \psi(t,x) + \kappa\|u\| \qquad \forall(t,x,u)$$

and where $\kappa < 1$ is known and the bounding function ψ: $\mathbb{R}\times X \to [0,\infty)$ is known and continuous.

3.4 Feedback stabilization of a differential inclusion system

Before formulating the stabilization problem, a class of ***generalized*** feedback controls is identified.

Defn. 3.4.1: A multifunction F: $\mathbb{R}\times X \rightrightarrows U$ is a ***generalized feedback*** if F is upper semicontinuous with nonempty, convex and compact values.

A scalar example of a generalized feedback, defined on $\mathbb{R}$ and containing continuous selections as well as discontinuous selections, is the multifunction

$$x \mapsto Q(x) := \begin{cases} \{-1\}, & x < -\varepsilon \\ [-1,1], & |x| \leq \varepsilon \\ \{1\}, & x > \varepsilon, \end{cases}$$

where $\varepsilon \in [0,\infty)$. To verify that Q is a suitable candidate for a generalized feedback, note that Q has nonempty, convex and compact values. To show that Q is upper semicontinuous proposition 1.5.3 is invoked: since $\overline{Q(\mathbb{R})} = [-1,1]$, which is compact, and the graph of Q is closed, it follows that Q is upper semicontinuous.

For $\varepsilon' > 0$, a particular example of a continuous selection is the function

$$x \mapsto f(x) := \begin{cases} -1, & x < -\varepsilon \\ x/\varepsilon, & |x| \leq \varepsilon \\ 1, & x > \varepsilon \end{cases}$$

which can be thought of as a continuous ε-approximation to the discontinuous signum function

$$x \mapsto \mathrm{sgn}(x) := \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \, ; \end{cases}$$

this being a discontinuous selection of the above multifunction. A control function of this type is used by Corless and Leitmann [21] to guarantee ultimate boundedness of all possible system responses within an arbitrarily small neighbourhood of the zero state for a particular system. This continuous control approximates a discontinuous control which guarantees global uniform asymptotic stability of the state origin for the given system (see chapter 4).

The basic problem is to determine a class of generalized feedbacks which renders a compact set W, with $0 \in W$, globally uniformly asymptotically stable for the feedback controlled differential inclusion system :

$$\dot{x}(t) \in Ax + BF(t,x) + G_F(t,x(t)) \,, \qquad x(t_0)=x_0, \tag{3.4.1}$$

where $G_F(t,x) := G(t,x,F(t,x)) = G_1(t,x) + B[G_2(t,x) + \kappa_0(G_3 \mathrm{o} F)(t,x)]$

and $(G_3 \mathrm{o} F)(t,x) = \bigcup_{u \in F(t,x)} G_3(u)$.

The proposed stabilizing generalized feedback is constructed following the approach taken in §2.5. A second hypothesis is required, *viz.*

A3.2: (A,B) is a stabilizable pair with rank(B) = m.

By assumption A3.2, there exists $K \in \mathbb{R}^{m \times n}$ such that the feedback $u(t) = Kx(t)$ stabilizes the nominal linear system. This feedback is then augmented by an appropriate multifunction to yield a generalized feedback $\boldsymbol{F}$ which bestows on a compact set W the stability property described above in the presence of uncertainty. Since $\bar{A} = A + BK$ is asymptotically stable, given a symmetric $Q > 0$, there exists a unique, symmetric $P > 0$ of the Lyapunov equation

$$P\bar{A} + \bar{A}^T P + Q = 0. \tag{3.4.2}$$

Multifunctions $\boldsymbol{P}_1: X \rightrightarrows \ker(B^T)$, $\boldsymbol{P}_2: X \rightrightarrows U$ and $\boldsymbol{D}: U \rightrightarrows U$ are now introduced, defined by

$x \mapsto \boldsymbol{P}_1(x) := \{ y \in \ker(B^T): \langle y, Px \rangle \geq 0 \}$,

$x \mapsto \boldsymbol{P}_2(x) := \{ y \in U : \langle y, B^T Px \rangle \geq 0 \}$ and

$$u \mapsto \boldsymbol{D}(u) := \begin{cases} \{\|u\|^{-1}u\}, & u \in U \backslash \varepsilon \bar{\mathcal{B}}_U \\ \\ \bar{\mathcal{B}}_U, & \text{otherwise}. \end{cases} \tag{3.4.3}$$

To ensure feedback stabilizability, an assumption on the residual uncertainty is now imposed.

A3.3: $\xi(\boldsymbol{G}_1(t,x) \cap \boldsymbol{P}_1(x)) \leq \kappa_1 \|x\| + \kappa_2 \quad \forall\ (t,x)$

where $\kappa_1, \kappa_2 \geq 0$ are known constants.

The proposed generalized feedback is given by the multifunction

$$(t,x) \mapsto \boldsymbol{F}(t,x) := Kx - \rho(t,x)\boldsymbol{D}(\rho(t,x)B^T Px) \tag{3.4.4a}$$

wherein $\rho: \mathbb{R} \times X \to \mathbb{R}_0^+$ is any continuous function satisfying

$$\rho(t,x) > \rho_0(t,x) := (1-\kappa_0)^{-1}[\kappa_0\|Kx\| + \xi(G_2(t,x)\cap P_2(x))] \ . \qquad (3.4.4b)$$

Clearly, F takes convex and compact values; moreover, the continuity of ρ and the upper semicontinuity of D ensure, by proposition 1.5.6, that F is also upper semicontinuous. Hence F qualifies as a generalized feedback.

Remarks:(*i*) The intersection $G_2(t,x)\cap P_2(x)$ is adopted in (3.4.4b) to economize on control gain by exploiting the possible occurrence of "stability enhancing" uncertainties.

(*ii*)Although continuous selections are to be chosen from the upper semicontinuous multifunction D, it is noted that there exist discontinuous selections. Thus multifunctions of this type, with prescribed continuity property, are able to handle discontinuous feedback controls. This is invaluable when stabilization is considered using discontinuous feedback controls (see chapter 4).

Theorem 3.4.1:Let assumptions A3.1–3 hold, with

$$\kappa_1 < \tfrac{1}{2}\sigma_{min}(P^{-1}Q)\{\sigma_{min}(P)\sigma_{min}(P^{-1})\}^{\frac{1}{2}}.$$

Then the ellipsoid

$E^Q_\varepsilon = \{\ y \in X : \langle y, Py\rangle \leq (\gamma^Q_\varepsilon)^2\}$,

where $\gamma^Q_\varepsilon := (\kappa_2\{\sigma_{max}(P)\}^{\frac{1}{2}} + \{4\varepsilon\beta^2 + \kappa_2^2\sigma_{max}(P)\}^{\frac{1}{2}})/\beta^2$

and $\quad \beta^2 := \sigma_{min}(P^{-1}Q) - 2\kappa_1\sigma_{max}(P)$,

is globally uniformly asymptotically stable with respect to the controlled differential inclusion system (3.4.1) and the generalized feedback (3.4.4).

Proof:To establish existence of local solutions for the feedback controlled system (3.4.1), with F given by (3.4.3), the hypotheses of proposition 3.2.1 must be confirmed. Let

$$H_F(t,x) := Ax + BF(t,x) + G_F(t,x) \quad \forall\ (t,x).$$

Since F is upper semicontinuous with compact values, it follows (from A3.1(*ii*), propositions 1.5.5 and 1.5.4) that $(t,x) \mapsto (G_3 oF)(t,x)$ is also upper semicontinuous with compact values. Since H_F is the sum of upper semicontinuous, compact-valued multifunctions, it is itself upper semicontinuous with compact values. As a consequence of hypothesis A3.1(*ii*),

$$G_2(t,x) + \kappa_0(G_3 oF)(t,x) = G_2(t,x) + \kappa_0\xi(F(t,x))\bar{B}_U$$

and, from proposition 1.5.1, is clear that $(G_2 + G_3 oF)(t,x)$ and, hence, $H_F(t,x)$ are convex. Thus, propositions 3.2.1 and 3.2.2 imply that, for each $(t_0,x_0) \in \mathbb{R}\times X$, there exists a maximal solution $x\colon [t_0,t_1) \to X$ of (3.4.1).
To establish that every maximal solution has interval of existence $[t_0,\infty)$, the behaviour of the function $V\colon X \to [0,\infty)$, defined by $V(x) := \langle x, Px\rangle$, is examined along all maximal solutions of (3.4.1). For each maximal solution $x\colon [t_0,t_1) \to X$ $\dot{V}(x(t)) \in \mathcal{V}(t,x(t))$ for almost all t, where $\mathcal{V}(t,x) := \{\ 2\langle Px, y\rangle : y \in H_F(t,x)\ \}$. Using (3.3.2), (3.4.2) and (3.4.4a),

$$\mathcal{V}(t,x) = -\langle x, Qx\rangle + \{\ 2\langle Px, y\rangle : y \in (GoF)(t,x) - \rho(t,x)BD(\rho(t,x)B^TPx)\ \},$$

whence, in view of A3.1, (3.4.3) and (3.4.4b),

$$\mathcal{V}(t,x) \leq -\langle x, Qx\rangle + 4\varepsilon + \{\ 2\langle Px, y\rangle : y \in G_1(t,x)\ \}.$$

Finally, it follows from hypothesis A3.3 that

$$\max\mathcal{V}(t,x) \leq -\langle x, Qx\rangle + 4\varepsilon + 2\|Px\|(\kappa_1\|x\| + \kappa_2). \tag{3.4.5}$$

Compare this result with the corresponding result obtained in chapter 2, *viz.*

(2.5.10). Thus, applying the same arguments used in theorem 2.5.1,

$$\max V(t,x) \leq -(\beta\{V(x(t))\}^{\frac{1}{2}} - \kappa_2\beta^{-1}\{\sigma_{\max}(P)\}^{\frac{1}{2}})^2 + 4\varepsilon + \kappa_2^2\sigma_{\max}(P)/\beta^2 \qquad (3.4.6)$$

where $\beta^2 := \sigma_{\min}(P^{-1}Q) - 2\kappa_1\sigma_{\max}(P)$, and hence, using proposition 3.2.3, (3.4.1) has indefinite continuation of solutions. Moreover, as a consequence of lemma 3.2.1 and equation (3.4.6), the ellipsoid

$E_\varepsilon^Q := \{ y \in X : \langle y, Py \rangle \leq (\gamma_\varepsilon^Q)^2 \}$, where

$\gamma_\varepsilon^Q := (\kappa_2\{\sigma_{\max}(P)\}^{\frac{1}{2}} + \{4\varepsilon\beta^2 + \kappa_2^2\sigma_{\max}(P)\}^{\frac{1}{2}})/\beta^2$,

is globally uniformly asymptotically stable.

□

Corollary 3.4.1: For any feedback-controlled differential inclusion system of the form (3.4.1), such that A3.1-3 hold with $G_1 = \{0\}$, given any $r > 0$, there exists a generalized feedback F, satisfying (3.4.4), such that $r\bar{B}_X$ is globally uniformly finite-time stable.

Proof: Identical to that of corollary 2.5.1.

□

As an illustration of the above theory, consider a system with state space form :

$$\dot{x}_1(t) = ax_1(t) + x_2(t)$$
$$\dot{x}_2(t) = bx_1^3(t) + u(t) + g(u(t))$$

where the uncertain parameters a, b satisfy $|a| \leq \tilde{a}$ ($\tilde{a} > 0$), $0 < b \leq \tilde{b}$, and g(u) is the uncertainty in the input u and is bounded by $\|g(u)\| \leq \kappa\|u\|$ with $0 < \kappa < 1$ known. Here, A and B can be identified as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and it is clear that for this system all the uncertainty is ***not*** matched. The residual uncertainty is $G_1(x) = \{ ax_1 : |a| \leq \tilde{a} \}$ whilst, for the matched uncertainty, $G_2(x) = \{ bx_1^3 : 0 \leq b \leq \tilde{b} \}$.

Hypothesis A3.2 is satisfied, since (A,B) is a controllable pair, and hence there exists $K \in R^{1\times 2}$ such that A + BK is asymptotically stable. For example, $K = [-2 \quad -3]$ suffices, in which case

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

and $\sigma(\bar{A}) \subset \mathbb{C}^-$.

$$P = \tfrac{1}{4}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

is the unique, positive-definite, symmetric solution of the Lyapunov equation (3.4.2) with Q = I.

Note that $\xi(G_1(x) \cap P_1(x)) \leq \tilde{a}\|x\| \quad \forall x \in R^2$ and hence A3.1(i) is satisfied with $\kappa_1 = \tilde{a}$ and $\kappa_2 = 0$.

$B^T P x = \tfrac{1}{4}(x_1 + x_2), \qquad P_2(x) = \{ y \in R : (x_1+x_2)y \geq 0 \}$,

i.e.

$$P_2(x) = \begin{cases} (-\infty, 0], & x_1+x_2 < 0 \\ R, & x_1+x_2 = 0 \\ [0,\infty), & x_1+x_2 > 0, \end{cases}$$

and therefore the function $x \mapsto \xi(G_2(x) \cap P_2(x))$ reduces to the function

$$\tilde{f}: x \mapsto \begin{cases} 0 & ; \quad x_1+x_2 > 0,\ x_1 < 0 \text{ or } x_1+x_2 < 0,\ x_1 > 0 \\ |f(x_1)| & ; \qquad \text{otherwise} \end{cases}$$

where f: $x_1 \mapsto \tilde{b}x_1^3$. Since $\sigma_{\min}(P) = \frac{1}{4}(3-\sqrt{5})$ and $\sigma_{\min}(P^{-1}) = 3-\sqrt{5}$, it follows, by theorem 3.4.1, that if $\tilde{a} < \frac{1}{8}(3-\sqrt{5})^2$ then the ellipsoid

$\{ x \in R^2: \ \langle x, Px \rangle < 4\varepsilon/\beta^2 \}$, where $\beta^2 = 3-\sqrt{5}-\tilde{a}(3+\sqrt{5})/2$,

is globally uniformly asymptotically stable using the generalized feedback control $x \mapsto F(x) = -2x_1 - 3x_2 - \rho(x)D(\rho(x)(x_1+x_2)/4)$, where ρ satisfies

$$\rho(x) > \rho_0(x) := (1-\kappa)^{-1}[\kappa\{4x_1^2 + 9x_2^2\}^{\frac{1}{2}} + \tilde{f}(x)]$$

and D is defined by (3.4.3). In particular, a stabilizing continuous control is

$$t \mapsto u(t) = -2x_1 - 3x_2 - \rho_0(x)(x_1+x_2)\begin{cases} |x_1+x_2|^{-1}, & \frac{\rho_0(x)}{4}|x_1+x_2| > \varepsilon \\ \frac{\rho_0(x)}{4\varepsilon}, & \text{otherwise.} \end{cases}$$

3.5 Model-following extension revisited

In this section the model-following example, described in §2.6, is considered in the context of differential inclusions. This section is based on the paper by Ryan [65] which extends the results of §2.6 to the case where a nonlinear differential inclusion system to be emulated by (3.4.1) is specified *a priori* as

$$M^*: \ \dot{z}(t) = A^*z(t) + B^*w(t) + g^*(t,z(t),w(t)), \quad z(t) \in X,\ w(t) \in W(t) \quad (3.5.1)$$

where the triple (A^*,B^*,g^*) is assumed known and g^* is continuous. Here, the function g^* depends on the instantaneous values of the model input which distinguishes this problem from that considered by Ryan [65]. The model input (a selection from W) is unknown, but it is assumed that $t \mapsto W(t) \subset \mathbb{R}^p$ is a known continuous multifunction with nonempty, convex and compact values, i.e. only W is available instead of exact model input information. In effect, system (3.5.1) can be regarded as an uncertain model. In order to follow this model an estimate of the input to the model is required, i.e. a continuous selection of W is required. However, since the elements of W are unknown, the following technique is used to extract a continuous selection. The basic idea is to determine w* such that

$$W(t) \subset w*(t) + r(t)\bar{B}_{\mathbb{R}^p},$$

where $w^*(t) := c(W(t))$ is the ***Chebishev centre*** of $W(t)$ (see Aubin and Cellina [7]) and r is continuous.

Remark:The selection $w*(\cdot)$ can be regarded as an estimate of the true (but unknown) model input $t \mapsto w(t) \in W(t)$.

Defn.3.5.1:The ***Chebishev radius*** $r_c(t)$ of $W(t) \subset \mathbb{R}^p$ is defined by

$$r_c(t) := \inf \{ r: W(t) \subset w + r\bar{B}_{\mathbb{R}^p} \text{ for some } w \in \mathbb{R}^p \}.$$

It can be shown (Aubin and Cellina [7], chapter 1, §8, proposition 2) that the set $\bigcap_{r>r_c(t)} \{ w: W(t) \subset w + rB_{\mathbb{R}^p} \}$ consists of a single point, $c(W(t))$, called the ***Chebishev centre*** of $W(t)$ and, moreover,

$$W(t) \subset c(W(t)) + r_c(t)B_{\mathbb{R}^p}.$$

Furthermore, the function $r_c(\cdot)$ is continuous and, as a consequence of a theorem by Aubin and Cellina [7] (chapter 1, §8, theorem 1), $c(W(\cdot))$ is continuous.

Remark: $r_c(t)$ is a bound for the input error $w(t)-w^*(t) := w(t)-c(W(t))$, i.e.

$$\| w(t)-w^*(t) \| < r_c(t).$$

The corresponding feasibility hypothesis for the model is :

A3.4:(*i*) Global existence and uniqueness of model solutions:

For each $(t_0,z_0,w) \in \mathbb{R}\times X\times L^\infty(\mathbb{R};\mathbb{R}^p)$, with $w(t) \in W(t)$, there exists a unique absolutely continuous function $z: [t_0,\infty) \to X$ satisfying (3.5.1) a.e., with $z(t_0) = z_0$.

(*ii*) Model-following conditions:

(a) $\mathrm{im}(A^*-A) \subset \mathrm{im}(B)$

(b) $\mathrm{im}(B^*) \subset \mathrm{im}(B)$

(c) $\mathrm{im}(g^*) \subset \mathrm{im}(B)$.

All the assumptions of §3.3 and 3.4 are assumed to hold and all uncertainty in the system is assumed to be matched (i.e. $G_1 \equiv \{0\}$) for the feedback-controlled system S^*:

$$\left.\begin{array}{l} \dot{x}(t) \in Ax(t) + BF(t,x(t),z(t)) + G_F(t,x(t),z(t)), \\ x(t_0) = x_0, \\ z(\cdot) \text{ a solution of the model } M^*, \end{array}\right\} \qquad (3.5.2)$$

where $G_F(t,x,z) := G(t,x,F(t,x,z)) = \bigcup_{u\in F(t,x,z)} G(t,x,u)$.

It is supposed that definition 2.6.1 applies to the differential inclusion systems M^* and S^*, defined by (3.5.1) and (3.5.2) respectively.

As a consequence of the model-following conditions, $A^*-A = BK^*$, $B^* = BL^*$, $g^* = Bg_2^*$, where K^*, L^*, g_2^* are defined in (2.6.6), (2.6.7) and (2.6.8), respectively.

The proposed generalized feedback for model-following is the multifunction $F^*: \mathbb{R}\times X\times X \rightrightarrows U$ defined by

$$F^*(t,x,z) := \Psi(t,x,z) + \mathrm{Þ}_\varepsilon(t,x,z) \tag{3.5.3}$$

where $\Psi(t,x,z) := K(x-z) + K^*z + L^*w^*(t)$ (3.5.4)

and, for $\varepsilon \in (0,\infty)$,

$$\mathrm{Þ}_\varepsilon(t,x,z) := -\rho^*(t,x,z)D(\rho^*(t,x,z)B^TP(x-z)), \tag{3.5.5}$$

with D defined by (3.4.3) and ρ^* is any continuous function, defined on $\mathbb{R}\times X\times X$, satisfying

$$\rho^*(t,x,z) > \rho_0^*(t,x,z) := (1-\kappa_0)^{-1}[\ \kappa_0\|K(x-z)+K^*z+L^*w^*(t)\| +$$
$$\|L^*\|r_c(t) + \xi(G_2^*(t,x)) + \xi(G_2(t,z)\cap P_2(x-z))] \tag{3.5.6}$$

in which P_2 is defined as in §3.4 and the multifunction $G_2^*: \mathbb{R}\times X \rightrightarrows U$ is defined by

$$(t,z) \mapsto G_2^*(t,z) := \{\ g_2^*(t,z,w)\ :\ w \in W(t)\}.$$

<u>Theorem 3.5.1</u>: Suppose A3.1-2, A3.4 hold and $G_1 \equiv \{0\}$. For F^*, defined by (3.5.3)-(3.5.6), the feedback-controlled system S^* follows the model M^* to within any compact set containing the ellipsoid

$$\{\ y \in X\ :\ \langle y,\ Py\rangle < 4\varepsilon\sigma_{max}(Q^{-1}P)\ \}.$$

<u>Proof</u>: Introducing the error $e(t) = x(t) - z(t)$, then, in view of (3.5.1)-(3.5.4),

$$\dot{e}(t) \in G_\varepsilon^*(t,e)\ ,$$

where, for each (t_0,z_0,w) and $w \in W$, the multifunction G_ε^* is defined by

$$(t,e) \mapsto G^*_\varepsilon(t,e) := \bar{A}e + B[\mathrm{p}_\varepsilon(t,z(t)+e,z(t)) + G_2(t,z(t)+e)$$
$$+ \kappa_0(G_3 o F^*)(t,z(t)+e,z(t)) - L^* r^*_c(t) - \bar{g}^*_2(t,z(t))],$$

with $z(\cdot)$ satisfying M^*, $r^*_c(t) \in r_c(t)\bar{B}_{\mathbb{R}^p}$ and $\bar{g}^*_2(t,z) \in G^*_2(t,z)$.

It is easily verified that G^*_ε is u.s.c. with convex and compact values. Hence, as a consequence of theorem 3.4.1, relating to the system

$$\dot{e}(t) \in G^*_\varepsilon(t,e(t)) , \qquad e(t_0) = x_0 - z_0,$$

the set $\{ y \in X : \langle y, Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P) \}$ is globally uniformly finite-time stable (see theorem 2.6.1).

□

Corollary 3.5.1: Suppose $G_1 \equiv \{0\}$ and assumptions A3.1-2 and A3.4 hold. Given any neighbourhood N of the origin in X, there exists a control F^*, defined by (3.5.3)-(3.5.6), such that the feedback-controlled system S^* follows the model M^* to within N.

Remark: A relaxed version of the model-following conditions is also applicable to the differential inclusion system S^* (see §2.6). In this case A3.4(*ii*) is replaced by :

A3.4(*ii*)*: For each $(t_0,z_0,w) \in \mathbb{R}\times X\times L^\infty(\mathbb{R};\mathbb{R}^p)$, with $w(t) \in W(t)$, the function $\zeta^*: [t_0,\infty) \to X$ given (a.e.) by

$\zeta^*(t) := \Pi_p[(A^* - A)z(t) + B^*w(t) + g^*(t,z(t),w(t))]$,

with $z(t) = Z(t,t_0,z_0,w)$, is essentially bounded.

Theorem 3.5.2: Suppose assumptions A3.1-2, A3.4(*i*) and A3.4(*ii*)* hold, and $G_1 \equiv \{0\}$. Then the generalized feedback F^*, given by (3.5.3)-

(3.5.6), is such that the feedback–controlled system S^* follows the model M^* to within every neighbourhood containing the ellipsoid $E^{\beta}_{\varepsilon} := \{ y \in X : \langle y, Py \rangle \leq (\gamma^{\beta}_{\varepsilon})^2 \}$, where
$\gamma^{\beta}_{\varepsilon} := \beta\sigma_{max}(Q^{-1}P) + [\{\beta\sigma_{max}(Q^{-1}P)\}^2 + 4\varepsilon\sigma_{max}(Q^{-1}P)]^{\frac{1}{2}}$.

Corollary 3.5.2: Let assumptions A3.1–2, A3.4(i), A3.4(ii)* hold. For any $\delta > 0$, the generalized feedback F^* is such that the feedback–controlled system S^* follows M^* to within the set $N(E^{\beta}_{0}, \delta)$.

3.6 Tracking problem for a differential inclusion system

The results of §3.4 can also be extended to the problem of tracking a prescribed function $z(\cdot)$. Here it is assumed that all uncertainty in the system :

$$\dot{x}(t) \in Ax(t) + BF(t,x(t),z(t)) + G(t,x(t),F(t,x(t),z(t))), \quad x(t_0)=x_0, \tag{3.6.1}$$

is matched and assumptions A3.1–2 hold. For this section, definition 2.7.1 is assumed to apply to the differential inclusion system (3.6.1).
To ensure feasibility of the motion z to be tracked, the following hypothesis is required :

A3.5: There exists a function $\theta \in L^{\infty}(R;R^m)$ such that

$$\dot{z}(t) = Az(t) + B\theta(t) \quad \text{a.e..}$$

The proposed generalized feedback for tracking z is given by

$$(t,x) \mapsto F(t,x) := \psi(t,x) + \mathrm{p}_{\varepsilon}(t,x) \tag{3.6.2}$$

$$\text{where } \psi(t,x) := K(x-z(t)) + \theta(t) \tag{3.6.3}$$

and, for $\varepsilon \in (0,\infty)$,

$$p_\varepsilon(t,x) := -\rho(t,x)D(\rho(t,x)B^TP(x-z(t))), \tag{3.6.4}$$

with D defined by (3.4.3) and ρ is any continuous function satisfying

$$\rho(t,x) > \rho_0(t,x) := (1-\kappa_0)^{-1}[\xi(G_2(t,x)\cap P_2(x-z(t))) + \kappa_0\|\psi(t,x)\|], \tag{3.6.5}$$

in which P_2 is defined as in §3.4.

<u>Theorem 3.6.1</u>: Under assumptions A3.1–2 and A3.5, the feedback-controlled system (3.6.1) (with $G_1 \equiv \{0\}$) and (3.6.2)–(3.6.5) tracks an absolutely continuous function $z(\cdot)$ to within any neighbourhood of the ellipsoid

$$\{ y \in X : \langle y, Py\rangle \leq 4\varepsilon\sigma_{max}(Q^{-1}P) \}.$$

<u>Corollary 3.6.1</u>: Assuming the conditions stated in theorem 3.6.1 hold, then, given any neighbourhood N of the origin in X, the generalized feedback F, given by (3.6.2)–(3.6.5), is such that the feedback-controlled system (3.6.1) tracks $z(\cdot)$ to within N.

Given the relaxed feasibility of tracking assumption:

A3.5*: The function $\zeta: [t_0,\infty) \to X$, defined (a.e.) by

$$\zeta(t) := \Pi_p[\dot{z}(t) - Az(t)]$$

is essentially bounded.

the following results are obtained:

Theorem 3.6.2: Suppose assumptions A3.1–2, A3.5* hold for system (3.6.1) (with $G_1 \equiv \{0\}$) and an absolutely continuous function z: $\mathbb{R} \to X$. Let θ: $\mathbb{R} \to \mathbb{R}^m$ satisfy a.e. $\theta(t) = \Pi[\dot{z}(t) - Az(t)]$. Then the generalized feedback F, given by (3.6.2)–(3.6.5), renders the ellipsoid E_ε^β (defined in theorem 3.5.2) globally uniformly asymptotically stable for system (3.6.1), where β satisfies $\{<\zeta(t), P\zeta(t)>\}^{\frac{1}{2}} < \beta$ a.e..

Corollary 3.6.2: Let assumptions A3.1–2, A3.5* hold. Then, given any neighbourhood N of the origin in X, the generalized feedback F (see (3.6.2)–(3.6.5)) is such that the system (3.6.1) (with $G_1 \equiv \{0\}$) tracks z to within N.

3.7 Application to a cylindrical robot

Deterministic control of uncertain dynamical systems has been applied to robotics, see, for example, Ambrosino, Celentano, and Garofalo [2], Corless, Leitmann, and Ryan [22], Ryan, Leitmann, and Corless [68], Young [89]. In particular, in the application to robotics, systems modelled by differential inclusions have been investigated by Paden and Sastry [59], and Slotine and Sastry [76]. Here, in this section, a continuous robust feedback control, of the form described in §3.6, is applied to the tracking problem for a cylindrical robot (carrying variable loads), with three controlled degrees of freedom. For a description of a cylindrical robot, see Freund [26].

It is desired that the robot track a prescribed path in the sense that (w.r.t. the position error), given any compact set of the zero state of the error system, a nonlinear continuous feedback control can be designed so that the error state enters the compact set in finite time and remains within it thereafter.

A simplified diagramatic representation of a cylindrical robot is shown below.

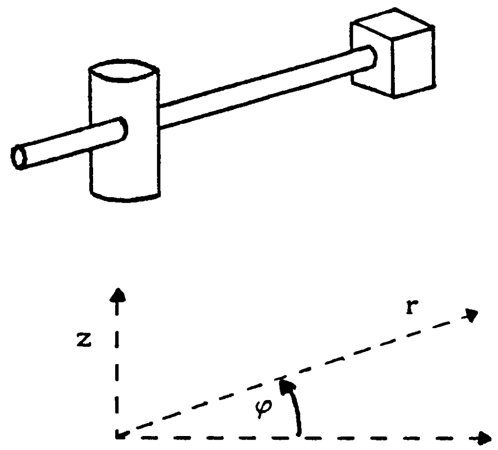

Fig. 4: Simplified illustration of a cylindrical robot and its associated degrees of freedom.

The cylindrical robot has a rotational joint and a translational joint in the (r,φ) plane. Furthermore, the arm of the robot can be moved vertically. Thus, in total, there are three degrees of freedom. The first degree of freedom is the vertical translation z which is driven by a force f_z. The second degree of freedom is the azimuth rotation φ and is driven by a torque f_φ. The third degree of freedom is the radial translation r, measured between the azimuth axis and the centre of gravity of the arm of the robot (of length R and which has mass m_a), and is driven by the force f_r. For simplicity it is assumed that the "hand" of the robot and the load are taken to be a point mass, m, which is regarded as the uncertain element in the model and is assumed to be constant, lying within known bounds, *viz.* $\underline{m} \leq m \leq \bar{m}$. A *nominal* load mass of m_0 is taken to be the harmonic mean of $\underline{m}$ and $\bar{m}$, i.e. m_0 satisfies

$$m_0^{-1} = \tfrac{1}{2}(\underline{m}^{-1} + \bar{m}^{-1}) \qquad (3.7.1)$$

The mass and radius of the cylindrical column are denoted by m^* and r^*, respectively. The equations of motion for the cylindrical robot are derived from the Lagrangian :

$$L := \tfrac{1}{2}\tilde{m}(m)\dot{r}^2 + \tfrac{1}{2}(J + j(r,m))\dot{\varphi}^2 + \tfrac{1}{2}M(m)\dot{z}^2 - M(m)gz,$$

where $J := \frac{1}{2}m^* r^{*2} + m_a R^2/3$, $j(r,m) := \tilde{m}(m)r^2 - m_a rR$ are the moments of inertia of the arm and cylindrical column, respectively, $\tilde{m}(m) = m_a + m$ is the combined mass of the arm and the load, $M(m) := m^* + \tilde{m}(m)$ is the total mass of the system, and g (assumed to be constant) is the acceleration due to gravity. Neglecting friction, the equations of motion for the cylindrical robot are of the form :

$$\tilde{m}(m)\ddot{r} - \tfrac{1}{2}\frac{\partial j}{\partial r}(r,m)\dot{\varphi}^2 = f_r$$

$$(J + j(r,m))\ddot{\varphi} + \frac{\partial j}{\partial r}(r,m)\dot{r}\dot{\varphi} = f_\varphi$$

$$M(m)(\ddot{z} + g) = f_z \quad .$$

With state variables :

$$x_1(t) = r(t), \qquad x_2(t) = \varphi(t), \qquad x_3(t) = z(t),$$

$$x_4(t) = \dot{r}(t), \qquad x_5(t) = \dot{\varphi}(t), \qquad x_6(t) = \dot{z}(t),$$

and control variables :

$$u_1(t) = \{m_a + m_0\}^{-1} f_r(t), \qquad u_2(t) = \{J + j(x_1, m_0)\}^{-1} f_\varphi(t), \qquad u_3(t) = \{M(m_0)\}^{-1} f_z(t),$$

the system can be formulated as the differential inclusion system :

$$\dot{x}(t) \in Ax(t) + B\{E(x(t)) + H(x(t))u(t)\} \ ,$$

where $x(t) = [x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)\ x_5(t)\ x_6(t)]^T$,

$$A = \begin{bmatrix} O & I \\ O & O \end{bmatrix}, \qquad B = \begin{bmatrix} O \\ I \end{bmatrix} \tag{3.7.2}$$

($A, B \in R^{6\times6}$ and $O, I \in R^{3\times3}$ are the zero and identity matrices, respectively.),

$$E(x) = \left\{ \begin{bmatrix} \frac{1}{2}(\tilde{m}(m))^{-1}\frac{\partial j}{\partial x_1}(x_1,m)x_5{}^2 \\ -(J+j(x_1,m))^{-1}\frac{\partial j}{\partial x_1}(x_1,m)x_4x_5 \\ -g \end{bmatrix} \; : \; m \in [\underline{m},\bar{m}] \right\} \tag{3.7.3}$$

and $H(x) = \{ \Lambda(x,m) : m \in [\underline{m},\bar{m}] \}$, with

$$\Lambda(x,m) := \begin{bmatrix} (\tilde{m}(m))^{-1}\tilde{m}(m_0) & 0 & 0 \\ 0 & (J+j(x_1,m))^{-1}(J+j(x_1,m_0)) & 0 \\ 0 & 0 & (M(m))^{-1}M(m_0) \end{bmatrix}$$

It is supposed that the robot tracks a path defined in terms of known functions $t \mapsto p_i(t)$ $(i = 1, 2, 3)$, with $p_i(t) \in C^2([t_0,\tau])$ for each i, by

$$x_1(t) = p_1(t), \quad x_2(t) = p_2(t), \quad x_3(t) = p_3(t) \quad \text{for } t \in [t_0,\tau].$$

Let $p_4(t) = \dot{x}_1(t)$, $p_5(t) = \dot{x}_2(t)$ and $p_6(t) = \dot{x}_3(t)$ for $t \in [t_0,\tau]$.
Define $p(t) := [p_1(t)\ p_2(t)\ p_3(t)\ p_4(t)\ p_5(t)\ p_6(t)]^T$ and $e(t) := x(t) - p(t)$ be the deviation between the actual and reference trajectories, together with the deviation between corresponding velocity components. The error function $t \mapsto e(t)$ satisfies

$$\dot{e}(t) \in Ae(t) + B(E(e(t)+p(t)) + H(e(t)+p(t))u(t) + Ap(t) - \dot{p}(t),$$

with $e(t_0) = e_0 := x(t_0) - p(t_0)$, which can be rewritten as

$$\dot{e}(t) \in Ae(t) + Bu(t) + BG(t,e(t),u(t)) \;,\; e(t_0) = e_0, \tag{3.7.4}$$

where $G(t,e,u) := E(e+p(t)) - p^*(t) + (H(e+p(t)) - I)u$ and

$$p^*(t) := [\ddot{p}_1(t)\ \ddot{p}_2(t)\ \ddot{p}_3(t)]^T. \tag{3.7.5}$$

The multifunction $(t,e,u) \mapsto G(t,e,u)$ is a particular example of the multifunction

$$(t,e,u) \mapsto G_p(t,e,u) := E(e+p(t)) - p^*(t) + \kappa\|u\|\bar{B}_{R^3} \;, \tag{3.7.6}$$

where $\kappa := \sup_{m \in [\underline{m},\bar{m}]} \|\Lambda(x,m) - I\|$,

and $G_p(t,e,u)$ can easily be identified with $G_2(t,e) + \kappa_0 G_3(u)$ given in A3.1.

Note that

$\|\Lambda(x,m) - I\| = \max |\sigma(\Lambda - I)|$

$= \max\{|(\tilde{m}(m))^{-1}\tilde{m}(m_0)-1|, |(J+j(x_1,m))^{-1}(J+j(x_1,m_0))-1|, |(M(m))^{-1}M(m_0)-1|\}.$

Now $|(J+j(x_1,m))^{-1}(J+j(x_1,m_0))-1|$ attains a maximum value of

$$4J|m_0 - m|/(4J\tilde{m}(m) - m_a^2R^2)$$

and since

$$\frac{4J}{4J\tilde{m}(m)-m_a^2R^2} > \tilde{m}^{-1}(m) > M^{-1}(m)$$

it follows that $\|\Lambda(x,m) - I\| = 4J|m_0 - m|/(4J\tilde{m}(m) - m_a^2R^2)$.

Therefore,

$$\kappa = \sup_{m \in [\underline{m},\bar{m}]} \{4J|m_0 - m|/(4J(m + m_a) - m_a^2R^2)\}.$$

Using (3.7.1),

$$\kappa = 4J|m_0 - \bar{m}|/(4J(\bar{m} + m_a) - m_a^2R^2)$$

$$= 4J\bar{m}(\bar{m} - \underline{m})/\{(4J(\bar{m} + m_a) - m_a^2R^2)(\bar{m} + \underline{m})\} \;. \tag{3.7.7}$$

Since $4J - m_aR^2 > 0$ and $(\bar{m} - \underline{m})/(\bar{m} + \underline{m}) < 1$, it follows that $\kappa < 1$. Consider the system :

$$\dot{e}(t) \in Ae(t) + Bu(t) + BG_p(t,e,u) , \qquad e(t_0) = e_0,$$

with G_p defined by (3.7.3), (3.7.5)–(3.7.7). This system can be identified with the system defined by (3.3.1) and (3.3.2), whence it follows that all uncertainty in the system is matched. Hypothesis A3.1 (in §3.3) is satisfied with $G_1 = \{0\}$, $G_2 \equiv E - p^*$ and $\kappa_0 \equiv \kappa < 1$. $E(x)$, by definition, is upper semicontinuous with compact values. Since $E(x)$ has convex values , G_p satisfies hypothesis A3.1(*iii*) (of §3.3). Clearly (A,B) is a controllable pair and, choosing $K = -[2I \quad 3I]$, $u(t) = Ke(t)$ stabilizes the nominal linear system. Solving the Lyapunov equation (3.4.2),

$$P = \tfrac{1}{4}\begin{bmatrix} 5I & I \\ I & I \end{bmatrix} .$$

A particular example of a continuous stabilizing feedback (see (3.4.4)) is

$$u(t) = Ke(t) - \rho_0(t,e(t))\eta(t,e(t))\begin{cases} \|\eta(t,e(t))\|^{-1}, & \text{if } \|\eta(t,e(t))\|>\epsilon \\ \epsilon^{-1} & , \text{ otherwise}, \end{cases} \tag{3.7.8}$$

where $\eta(t,e) := \rho_0(t,e)B^TPe$,

$\rho_0(t,e) := (1-\kappa)^{-1}[\ \kappa\|Ke\| + \xi(G_2(t,e+p) \cap P_2(e+p))]$ and κ is given by (3.7.7).

Since $G_2(t,e+p) = E(e+p) - p^*(t)$, a suitable choice for $\rho_0(t,e)$ is

$$\rho_0(t,e) = (1-\kappa)^{-1}[\kappa\|Ke\| + g(t,e)],$$

where $(t,e) \mapsto g(t,e)$ is defined by

$$g(t,e) := \{(\tfrac{1}{2}\tilde{m}(\underline{m})^{-1}\frac{\partial j}{\partial x_1}(x_1,\overline{m})x_5^2 - \ddot{p}_1(t))^2$$

$$+ ((J + j(x_1,\underline{m}))^{-1}\frac{\partial j}{\partial x_1}(x_1,\overline{m})x_4x_5 + \ddot{p}_2(t))^2$$

$$+ (g + \ddot{p}_3(t))^2\}^{\frac{1}{2}} \Big|_{x=e+p}. \qquad (3.7.9)$$

Thus, using the continuous feedback control (3.7.8), the system (3.7.4) tracks p to within any given neighbourhood of the origin in R^6 (see theorem 3.6.1 and corollary 3.6.1).

Consider a simulation in which the mass of the robot arm is 5kg, the mass of the cylindrical column is 14kg, the length of the robot arm is 1m, and the total moment of inertia about the z axis is 2kgm^2. The mass of the uncertain payload lies within the range 1 – 4kg and, for this simulation, it is assumed that its mass is 3kg. It is desired that the robot arm tracks a straight line path from (0.8, 0.0, 1.0) to (0.0, –0.8, –0.5) in 3 seconds. The computed evolution of the trajectories for the controlled system (with parameter ε = 0.01), defining the path of the robot arm, is illustrated in figure 5 shown below, together with the error norm function $\|e(\cdot)\| = \|x(\cdot) - p(\cdot)\|$. In figure 6, for comparison purposes, the paths (AB, CD and EF) of the feedback–controlled system in the (y_1,y_2), (y_1,y_3) and (y_2,y_3) planes, with respect to the Cartesian coordinates (y_1,y_2,y_3), is illustrated together with the path of the robot arm generated by the nominal open loop control, $t \mapsto \hat{u}(t)$, which would generate the desired paths AB, CD, EF under the nominal load $\tilde{m}(m_0)$. Finally, the evolution of the controls is shown in figure 7.

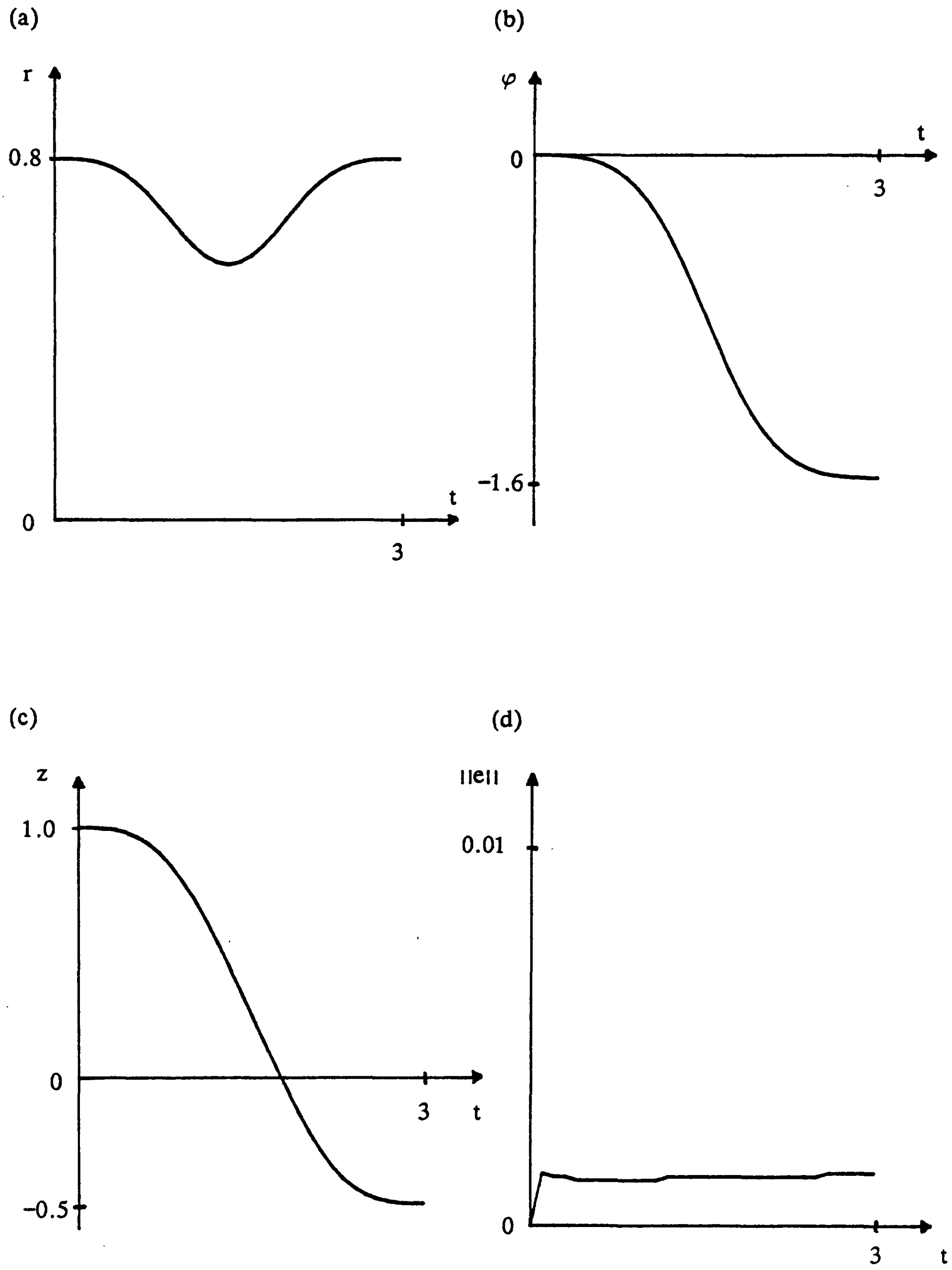

Fig. 5. System trajectories and tracking error norm.

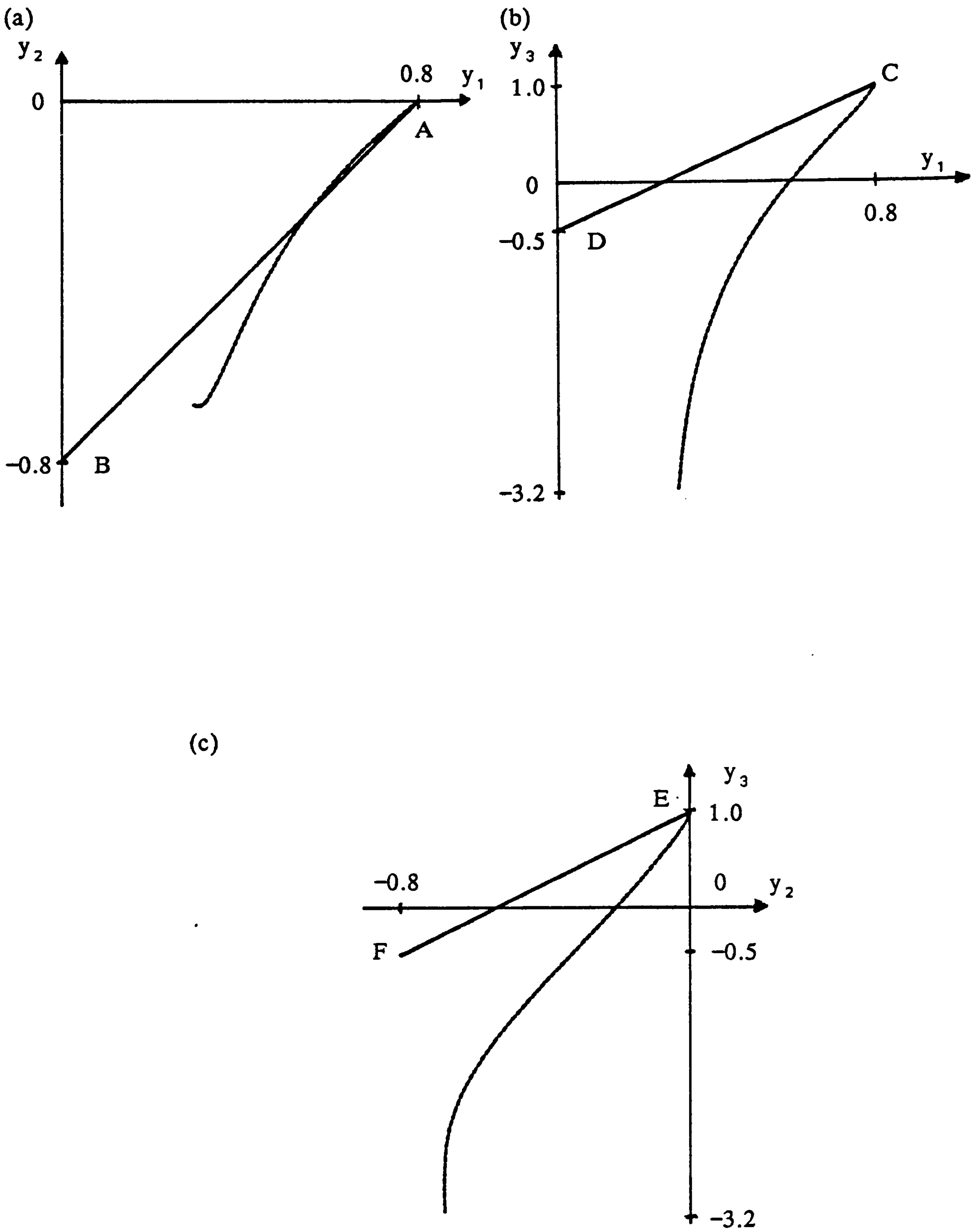

Fig. 6. Paths generated by feedback control u and nominal open loop control $\hat{u}$.

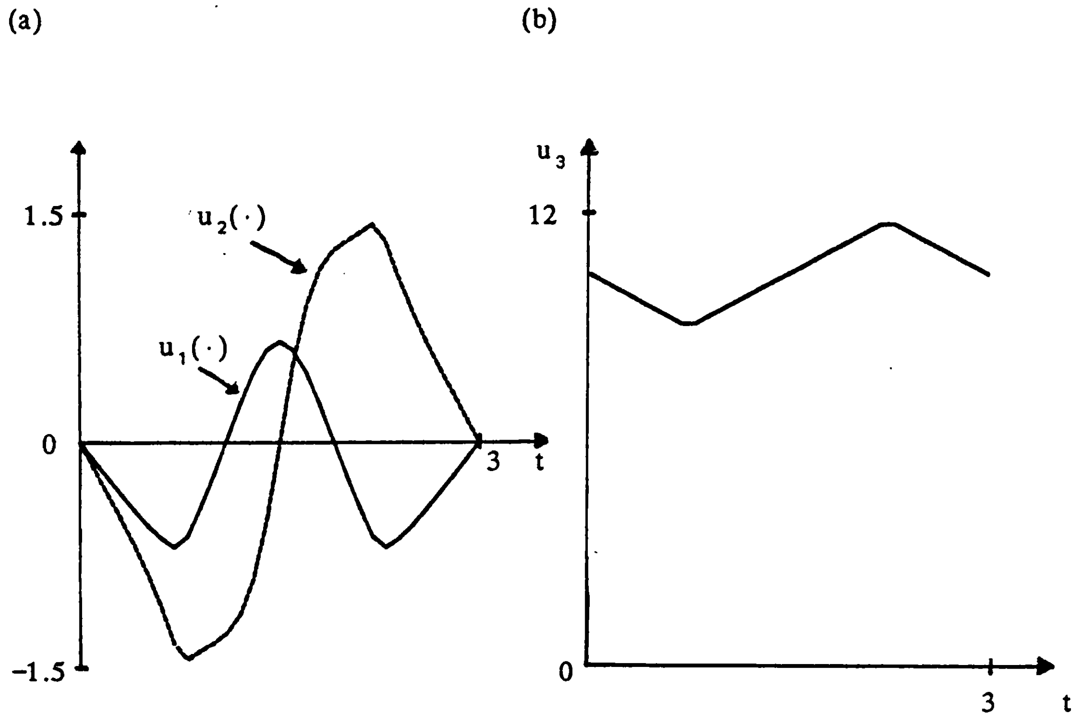

Fig. 7. Control histories.

4. FEEDBACK STABILIZATION OF UNCERTAIN SYSTEMS USING DISCONTINUOUS CONTROL

4.1 Introduction

The problem of finding a control function $t \mapsto u(t)$ for a system governed by

$$\dot{x}(t) = h(t,x(t),u(t)) \ , \qquad x(t) \in X, \ u(t) \in U, \tag{4.1.1}$$

where a given positive definite function $(t,x) \mapsto V(t,x)$ decreases along a trajectory, has been examined by many authors (see, for example, Kalman and Bertram [42]). If control synthesis is an objective, then, as mentioned in chapter 3, ***discontinuous*** feedback control, say $u(t) = D(t,x)$, is often a natural candidate. In this case, the resulting differential equation (modelling the feedback controlled system), is described by

$$\dot{x}(t) = h_D(t,x(t)) \ ; \qquad h_D(t,x) := h(t,x,D(t,x)), \tag{4.1.2}$$

where h_D has discontinuities in x and thus fails to satisfy the requisite hypotheses of classical theory. To overcome this difficulty, a problem formulation based on differental inclusions may be adopted. To address this particular problem some authors (see, for example, Filippov [24], Hermes [35], and Paden and Sastry [59]) have considered a differential inclusion of the form

$$\dot{x}(t) \in \bigcap_{\varepsilon>0} \ \bigcap_{\mu(N)=0} \overline{co}(h_D(t, \ (x + \varepsilon \bar{B}_X)\backslash N)) \ , \tag{4.1.3}$$

where $\overline{co}$ denotes the closed convex hull, $N \subset X$ is arbitrary and $\mu(N)$ denotes the Lebesgue measure of N. The trajectories of (4.1.3) are then expected to be "close" to the trajectories of the discontinuous system (4.1.2). This approach is not taken here since it presupposes that h_D is known precisely. Instead, since system uncertainty needs to be addressed, a differential inclusion system of the form

$$\dot{x}(t) \in H(t,x(t),u(t))$$

is considered, where the discontinuous feedback u(t) is embedded in a multifunction F, sufficiently "regular" in the sense that F qualifies as a generalized feedback (recall definition 3.4.1), so that the resulting differential inclusion system:

$$\dot{x}(t) \in H_F(t,x(t)), \qquad \text{where } H_F(t,x) = \bigcup_{u \in F(t,x)} H(t,x,u)$$

is such that H_F satisfies the conditions necessary for the existence of local solutions (see §3.2).

The use of discontinuous controls loses the advantage of smooth control action but gains in the fact that asymptotic stability of an arbitrary small neighbourhood of the state origin can be replaced by asymptotic stability of the state origin. Also, the adoption of a discontinuous feedback enables one to ulitize the theory of ***Variable Structure Systems*** (developed by, amongst others, Itkis [40] and Utkin [78],[79]). This theory has been adapted to achieve (global) stabilization of dynamical systems in the presence of parameter uncertainty and input disturbance (see Ryan [63], Slotine and Sastry [76], Young [89], and Zinober [92]). A variable structure system changes structure depending upon the state of a system. None of the structures, individually, are necessarily "stable", but, by synthesizing a suitable control, the combined variable structure system is "stable". The basis of this approach is a manifold W, in state space X, towards which all (local) trajectories are attracted and on which all motion is independent of uncertainty and disturbance. This is the ***invariance*** property of the manifold W. The control is designed so that W is attained in finite time along all trajectories of the feedback system; thereafter the motion is constrained to W and has the added advantage that it takes on prescribed dynamic behaviour of an ***ideal model***. In the literature on variable structure systems, W is known as a ***switching surface*** (since the control must be repeatedly switched between two values in order that a trajectory remains on W) and the invariance of W is referred to as the so-called ***sliding mode*** occuring on a switching surface. The idealized sliding mode is theoretical in nature. In practice, due to switching delays, neglected small time constants, e.t.c., "chattering" along switching surfaces occurs rather than ideal sliding. Slotine and Sastry [76], for example, show how continuous controls can be used to approximate the discontinuous control and overcome the problem of chattering.

In order that ***global*** attractivity of the manifold is achieved in the presence of uncertainty, structural conditions are imposed on the system so that the Lyapunov-based theory of Gutman, Leitmann, *et al* (see, for example, Gutman [31], Gutman and Palmor [33], and Leitmann [49]) can be invoked. The two deterministic theories can then be combined in a unifying approach. The approach taken here, in this chapter, is essentially that of Ryan and Corless [67], subsequently recast in a differential inclusion setting by Goodall and Ryan [28], [29].

The class of uncertain dynamical systems is described in §4.2, as well as constructing the manifold W. A linear component of the control is then designed so that the manifold W has the desired properties. In §4.3 the stabilization problem is discussed and a generalized feedback is proposed. The generalized feedback is obtained by augmenting the linear component obtained in §4.2 with a multifunction which contains nonlinear, discontinuous selections. With controls of this form, the manifold W is shown to be attractive (see §4.4) and stability of the system is investigated. Finally, in §4.5, the theory is illustrated by pursuing the robotics example previously studied in chapter 3.

4.2 The system and invariant manifold W

The system to be considered is of the form

$$\dot{x}(t) \in H(t,x(t),u(t)), \qquad x(t_0) = x_0 \qquad (4.2.1)$$

where H: $\mathbb{R}\times X\times U \rightrightarrows X$ is a known multifunction with non-empty values. For a given control function u: $[t_0,\infty) \to U$, x: $[t_0,\infty) \to X$ is deemed a solution or trajectory of (4.2.1) if it is absolutely continuous and satisfies (4.2.1) a.e. on $[t_0,\infty)$. To ensure existence of solutions, conditions of upper semicontinuity, and convexity and compactness of its values, will be imposed on H; moreover, to ensure feedback stabilizability (in a sense to be defined), the allowable class of multifunctions must be further restricted. The requisite hypotheses are contained in assumption A4.1 below and A4.2 introduced later.

A4.1: There exists a controllable pair $(A,B) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}$, with B of full rank $m < n$, such that the multifunction

$(t,x,u) \mapsto G(t,x,u) := H(t,x,u) - Ax - Bu$

has the following property:

there exist nonempty multifunctions G_1: $\mathbb{R}\times X \rightrightarrows \ker(B^T)$, G_2: $\mathbb{R}\times X \rightrightarrows U$ and a continuous function β: $\mathbb{R} \to [0,\kappa_0]$, $\kappa_0 < 1$, such that

(*i*) $\Pi_{\ker(B^T)} G(t,x,u) = G_1(t,x) \quad \forall (t,x,u)$;

(*ii*) $\Pi_{\mathrm{im}(B)} G(t,x,u) = B[G_2(t,x)+\beta(t)G_3(u)] \quad \forall (t,x,u)$, where G_3: $u \mapsto \|u\|\bar{B}_U$

(*iii*) G_1 and G_2 are upper semicontinuous with convex and compact values.

Remarks: Notice that the class of multifunctions, satisfying the hypotheses implicit in A4.1, is an extension of the class of multifunctions, described in chapter 3, satisfying A3.1. In this case κ_3, given in A3.1(*ii*), has been replaced by a time-varying function β.

The invariance property of a manifold W, alluded to in variable structure systems theory, is guaranteed if an (n−m)-dimensional manifold $W \subset X$ (on which system motion is governed by the dynamic equations of a prescibed linear model defined by (n−m) ***ideal model*** eigenvalues) is constructed in the following manner.

Let $L_1 \in \mathbb{R}^{(n-m)xn}$ be such that $\ker(L_1) = \mathrm{im}(B)$. Define $L_2 := (B^TB)^{-1}B^T$ and $L := [L_1 \quad L_2]^T$ with inverse $L^{-1} = R := [R_1 \quad B]$, then

$$LAR = \begin{bmatrix} L_1AR_1 & L_1AB \\ & \\ L_2AR_1 & L_2AB \end{bmatrix} \; ; \qquad LB = \begin{bmatrix} 0 \\ \\ I \end{bmatrix} .$$

Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\} \subset \mathbb{C}^-$ be the ideal model spectrum which implicitly determines $W \subset X$. Noting that (L_1AR_1, L_1AB) is a controllable pair (since, by A4.1, (A,B) is controllable), there exists $M \in \mathbb{R}^{mx(n-m)}$ such that

$$\sigma(L_1AR_1 + L_1ABM) = \sigma(L_1AM^*) = \Lambda,$$

where, for notational convenience, $M^* := R_1 + BM$. The (n−m)-dimensional manifold $W \subset X$ is now defined as

$$W := \ker(\bar{M}) , \qquad \bar{M} := L_2 - ML_1 . \tag{4.2.2}$$

A linear component, F, of the (generalized) feedback control for system (4.2.1) is now designed so that W is an invariant subspace of X for the nominal linear system :

$$\dot{x}(t) = Ax(t) + Bu(t),$$

i.e. for every $x \in W$, $Ax + Bu \in W$ with $u = Fx$. Also, the choice of F and the prescribed ideal model spectrum implicitly determine the spectrum of (A + BF).

Proposition 4.2.1: Let $C \in \mathbb{R}^{m \times m}$ be such that $\sigma(C) \subset \mathbb{C}^-$ and define

$$F := C\bar{M} - \bar{M}A. \tag{4.2.3}$$

Then (*i*) $\sigma(A+BF) \subset \mathbb{C}^-$

and (*ii*) $W \subset X$ is an $(A + BF)$–invariant subspace.

Proof: Let $T = [L_1 \quad \bar{M}]^T = [L_1 \quad L_2 - ML_1]^T$ with inverse $T^{-1} = [R_1 + BM \quad B]$,

$$\text{then} \qquad T(A+BF)T^{-1} = \begin{bmatrix} L_1AM^* & L_1AB \\ & \\ 0 & C \end{bmatrix}$$

whence

$$\sigma(A+BF) = \sigma(T(A+BF)T^{-1}) = \sigma(L_1AM^*) \cup \sigma(C) = \Lambda \cup \sigma(C) \subset \mathbb{C}^-,$$

which establishes (*i*). Now, noting that $\bar{M}BF = F = C\bar{M} - \bar{M}A$, one can conclude that $\bar{M}(A+BF)x = 0$ for every $x \in W = \ker(\bar{M})$, thereby proving (*ii*).

□

Remark: Clearly, there is considerable scope for judicious choice of the operators M and C. For example, it may be desirable to attempt to maximize the stability radius (see Hinrichsen and Pritchard [37],[38]) of

$$L_1AM^* = L_1AR_1 + L_1ABM$$

over the set of matrices $M \in \mathbb{R}^{m \times (n-m)}$ which assign the spectrum Λ to L_1AM^*.

In the ensuing section, the linear feedback operator F is augmented by an appropriate multifunction to yield a generalized feedback $\boldsymbol{F}$ which preserves the property of asymptotic stability of the origin in the presence of matched uncertainty.

4.3 Stabilization problem formulation and proposed generalized feedback

The fundamental problem to be studied is that of stabilization by feedback, viz. determine a (time-dependent) feedback strategy $(t,x) \mapsto F(t,x)$ such that, loosely speaking, all trajectories of (4.2.1), with $u \in F(t,x(t))$, exhibit "stable" behaviour. Here, the class of admissible feedbacks is taken to be the class of generalized feedbacks $F: \mathbb{R}\times X \rightrightarrows U$, defined in definition 3.4.1. A simple example of a generalized feedback in the scalar case is the multifunction $F = SGN$ (a set-valued signum function), given by, for $x \in \mathbb{R}$,

$$SGN: x \mapsto \begin{cases} \{1\}, & x > 0 \\ [-1,1], & x = 0 \\ \{-1\}, & x < 0 \end{cases}$$

i.e. a relay-type control function. An obvious discontinuous selection from $SGN(x)$ is

$$\text{sgn}: x \mapsto \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}.$$

Remark: In this chapter, the proposed generalized feedbacks are singleton-valued except on a set Γ_F of Lebesgue measure zero in $\mathbb{R}\times X$. The set Γ_F may be identified as a switching surface of control discontinuities.

The basic stabilization problem may now be stated as: determine a generalized feedback F such that the feedback-controlled differential inclusion :

$$\left.\begin{aligned} &\dot{x}(t) \in H_F(t,x(t)) \\ &H_F(t,x) := Ax+G_1(t,x)+BF(t,x)+B[G_2(t,x)+\beta(t)(G_3 o F)(t,x)] \end{aligned}\right\} \qquad (4.3.1)$$

exhibits a compact set (***positively invariant*** and containing the origin) which is globally uniformly asymptotically stable.

Remarks:(i)Assuming that H_F satisfies conditions necessary for the existence of a local solution x, which can be continued indefinitely, a set $S \subset X$ is ***positively invariant*** if $x \in S$ implies that the set of all x satisfying (4.3.1) is a subset of S for all $t_0 \in \mathbb{R}$.

(ii)Ideally, a generalized feedback F is sought which renders $S = \{0\}$ globally uniformly asymptotically stable; it will be shown (by construction) that such a feedback exists if $G_1 \equiv \{0\}$ (i.e. in the absence of residual uncertainty). If $G_1 \not\equiv \{0\}$, then, in essence, F is sought such that (4.3.1) exhibits a globally uniformly asymptotically stable compact set S (containing the origin) with acceptably small diameter.

(iii)If compact S is a globally uniformly asymptotically stable set for (4.3.1), then (4.3.1) is globally uniformly ultimately bounded (as described in Corless and Leitmann [21]) within every open set containing S.

In the construction of the generalized feedback F, the approach (discussed in Goodall and Ryan [28], [29]) is akin to that of Ryan and Corless [67] and draws on concepts from (a) variable structure systems theory (see Utkin [79]) and (b) Lyapunov-based theory (see Corless and Leitmann [21]). In particular, the concept of an invariant (n−m)-dimensional manifold $W \subset X$ is adopted from variable structure systems theory. The feedback is then designed (via Lyapunov based analysis) to ensure that: (i) W is invariant under H_F and the flow on W (in the absence of residual uncertainty) is equivalent to that of a linear system (***the ideal model***) with prescribed spectrum Λ; (ii) W is globally finite-time attractive in the sense that W is ultimately attained on every solution of (4.3.1). Thus the proposed feedback F, in the absence of residual uncertainty, guarantees that $\{0\}$ is a globally uniformly asymptotically stable set. It is important to note that, in addition to asymptotic stability, the proposed feedback F ensures that the flow (in the absence of residual uncertainty) exhibits other desired features, namely that ***all*** trajectories $x(\cdot)$ of (4.3.1) ultimately coincide with a trajectory of a linear system with prescribed spectrum Λ; more precisely, with L_1 and M^* defined as in §4.2, $x(t) = M^*w(t)$ for all t sufficiently large, where $w(\cdot)$ satisfies the linear equation $\dot{w} = (L_1AM^*)w$, with spectrum $\sigma(L_1AM^*) = \Lambda$. If residual uncertainty is present (i.e. if $G_1 \neq \{0\}$) then, under assumption A4.2 below, invariance and global finite-time attractivity of W are preserved. However, ideal model motion on W cannot be guaranteed; instead, global uniform asymptotic stability of a calculable compact set $S \subset W$ only is assured, the diameter of which is determined by bounds on the values of G_1.

Let $P_1 \in \mathbb{R}^{(n-m)x(n-m)}$ and $P_2 \in \mathbb{R}^{mxm}$ denote the unique positive-definite symmetric solutions of the Lyapunov equations

$$P_1L_1AM^* + (L_1AM^*)^TP_1 + I = 0 \tag{4.3.2}$$

and

$$P_2C + C^TP_2 + I = 0. \tag{4.3.3}$$

Define multifunctions P_1: $X \rightrightarrows \mathbb{R}^{n-m}$ and P_2: $X \rightrightarrows U$ by

$$x \mapsto P_1(x) := \{ \nu \in \mathbb{R}^{n-m} : \langle \nu, P_1L_1x \rangle \geq 0 \} \tag{4.3.4}$$

and

$$x \mapsto P_2(x) := \{ u \in U : \langle u, P_2\bar{M}x \rangle \geq 0 \}. \tag{4.3.5}$$

The final assumption on the residual uncertainty is now imposed.

A4.2 : $\xi(L_1 G_1(t,x) \cap P_1(x)) \leq \kappa_1 \|L_1 x\| + \kappa_2 \|\bar{M}x\| + \kappa_3 \qquad \forall(t,x)$

where κ_1, κ_2, and κ_3 are non-negative constants.

Define multifunction D: U $\rightrightarrows$ U as

$$u \mapsto D(u) := \begin{cases} \{\|u\|^{-1}u\}; & u \neq 0 \\ \bar{B}_U ; & u = 0 \end{cases}$$

which is clearly upper semicontinuous (D(u) is the generalized gradient, see Clarke [18], of $\|\cdot\|$: U $\to$ R$^+$ at u and can be regarded as an n-dimensional analogue of the multifunction *SGN*).

The proposed feedback is now given by the multifunction

$$(t,x) \mapsto F(t,x) := Fx - \rho(t,x) D(P_2\bar{M}x) \tag{4.3.6}$$

where F is given by (4.2.3) and ρ is any continuous functional on R×X satisfying $\rho(t,x) > \rho_0(t,x)$ with

$$\rho_0(t,x) := (1-\beta(t))^{-1}[\beta(t)\|Fx\| + \xi((G_2(t,x) - ML_1 G_1(t,x)) \cap P_2(x)) + \gamma] \tag{4.3.7}$$

where $\gamma > 0$ is a design parameter. Note that the continuity of ρ and the upper semicontinuity of D ensure (by proposition 1.5.6) that F is upper semicontinuous and clearly takes convex and compact values; thus, F qualifies as a generalized feedback. Note further that F is singleton-valued except on the set Γ_F = R×W.

Remark:The intersection $(G_2(t,x) - ML_1 G_1(t,x)) \cap P_2(x)$ is adopted in (4.3.7) in order to economize on control gain by exploiting the possible occurrence of "stability enhancing" uncertainties.

4.4 Attractivity of the manifold W and stabilization of the feedback system

First it is shown that the feedback controlled differential inclusion system, given by (4.3.1) and (4.3.6)–(4.3.7), exhibits the properties of existence and continuation of solutions and uniform boundedness of solutions.

Lemma 4.4.1: Under hypotheses A4.1 and A4.2, with $\kappa_1 < \frac{1}{2}\|P_1\|^{-1}$, the feedback–controlled system (4.3.1), with generalized feedback control F given by (4.3.6)–(4.3.7), exhibits properties of

(*i*) existence and continuation of solutions, and

(*ii*) uniform boundedness of solutions.

Proof: In order to establish the existence of at least one local solution x: $[t_0,\tau) \to X$ with $x(t_0) = x_0$, for each $(t_0,x_0) \in \mathbb{R}\times X$, proposition 3.2.1 is invoked. Thus it suffices to show that H_F, given by (4.3.1) and (4.3.6)–(4.3.7), satisfies the hypotheses of proposition 3.2.1. Since βG_3 and F are upper semicontinuous with compact values, it follows (by propositions 1.5.5 and 1.5.4) that $(t,x) \mapsto \beta(t)(G_3 \circ F)(t,x)$ is also upper semicontinuous with compact values; thus, H_F is the sum of upper semicontinuous multifunctions with compact values and hence is itself upper semicontinuous with compact values. Since βG_3 is convex valued and $F(t,x)$ is either a singleton or a closed ball (of radius $\rho(t,x)$), it follows that $(t,x) \mapsto \beta(t)(G_3 \circ F)(t,x)$ is convex valued; thus, H_F is the sum of convex valued multifunctions and hence is itself convex valued. For each (t_0,x_0), the existence of a local solution x: $[t_0,\tau) \to X$ with $x(t_0) = x_0$ now follows by proposition 3.2.1.

To establish that every such solution can be extended into a solution on $[t_0,\infty)$, consider the behaviour, along local solutions, of the function V: $X \to \mathbb{R}^+$ defined by

$$x \mapsto V(x) := \tfrac{1}{2}\langle x, [L_1^T \ \ \bar{M}^T] Q_\zeta [L_1 \ \ \bar{M}]^T x\rangle,$$

where $Q_\zeta := \begin{bmatrix} P_1 & 0 \\ & \\ 0 & \zeta P_2 \end{bmatrix}$ and $\zeta \in R^+$ (is to be specified).

Along each (local) solution $x\colon [t_0,\tau) \to X$

$$\dot{V}(x(t)) \in \mathcal{L}(t,x(t)) = \mathcal{L}_1(t,x(t)) + \mathcal{L}_2(t,x(t))$$

where $\quad \mathcal{L}_1(t,x) := \langle\langle L_1 \mathcal{H}_F(t,x),\ P_1 L_1 x\rangle\rangle \subset R$

$$\mathcal{L}_2(t,x) := \zeta\langle\langle \bar{M}\mathcal{H}_F(t,x),\ P_2 \bar{M}x\rangle\rangle \subset R.$$

Thus, if it can be shown that there exists $s \in R^+$ such that

$$\mathcal{L}(t,x) \cap [0,\infty) = \emptyset \qquad \forall (t,x) \in [t_0,\infty)\times(X\backslash s\bar{B}_X) \tag{4.4.1}$$

then it follows that every solution $x\colon [t_0,\tau) \to X$ with $x(t_0) = x_0$ evolves within the compact set

$$\{\ x\colon V(x) \le \max[V(x_0),\ \tfrac{1}{2}\|[L_1^T \ \ \bar{M}^T] Q_\zeta [L_1 \ \ \bar{M}]^T\|s^2]\ \}$$

and hence can be continued indefinitely. The property of uniform boundedness of solutions readily follows. It remains to establish the existence of $s \in R^+$ such that (4.4.1) holds. Now, using (4.3.2),

$$\mathcal{L}_1(t,x) = -\tfrac{1}{2}\|L_1 x\|^2 + \langle L_1 AB\bar{M}x,\ P_1 L_1 x\rangle + \langle\langle L_1 \mathcal{G}_1(t,x),\ P_1 L_1 x\rangle\rangle$$

whence, in view of A4.2,

$$\max\mathcal{L}_1(t,x) \le -\tfrac{1}{2}(1-2\kappa_1\|P_1\|)\|L_1 x\|^2 + (\|P_1 L_1 AB\| + \kappa_2\|P_1\|)\|L_1 x\|\|\bar{M}x\|$$
$$+ \kappa_3\|P_1\|\|L_1 x\|.$$

Using (4.3.3),

$$L_2(t,x) \leq \zeta[-\tfrac{1}{2}\|\bar{M}x\|^2 - \rho(t,x)\|P_2\bar{M}x\|$$
$$+ \langle\langle G_2(t,x) - ML_1G_1(t,x) + \beta(t)(G_3 o F)(t,x),\ P_2\bar{M}x\rangle\rangle]$$

whence, in view of (4.3.7),

$$\max L_2(t,x) \leq -\tfrac{1}{2}\zeta\|\bar{M}x\|^2.$$

Combining the above yields

$$\max L(t,x) \leq -\tfrac{1}{2}\langle[\|L_1x\| \quad \|\bar{M}x\|]^T, E_\zeta[\|L_1x\| \quad \|\bar{M}x\|]^T\rangle$$
$$+ \kappa_3\|P_1\|\|L_1x\| \qquad (4.4.2)$$

where

$$E_\zeta := \begin{bmatrix} 1-2\kappa_1\|P_1\| & -(\|P_1L_1AB\|+\kappa_2\|P_1\|) \\ -(\|P_1L_1AB\|+\kappa_2\|P_1\|) & \zeta \end{bmatrix}. \qquad (4.4.3)$$

By virtue of A4.2 and choosing ζ such that

$$\zeta > (1-2\kappa_1\|P_1\|)^{-1}[\|P_1L_1AB\| + \kappa_2\|P_1\|]^2 ,$$

the first term on the right hand side of (4.4.2) is a negative definite quadratic form in x, which clearly ensures the existence of $s \in \mathbb{R}^+$ such that (4.4.1) holds.

□

Corollary 4.4.1: If $\kappa_3 = 0$, then the zero state is a globally uniformly asymptotically stable equilibrium of system (4.3.1) and (4.3.6)–(4.3.7).

Proof: Note that, if $\kappa_3 = 0$, then (4.4.1) holds with $s = 0$.

□

To establish that every solution of the feedback controlled differential inclusion system attains the subspace W in finite time and thereafter remains in W, the behaviour of the function V_2: $X \to R^+$, $x \mapsto V_2(x) := \frac{1}{2}\langle x, \bar{M}^T P_2 \bar{M} x\rangle$, along solutions is considered.

Lemma 4.4.2: For each $(t_0, x_0) \in R \times X$, the manifold W is attained in

$$t_f < \gamma^{-1}[2\|P_2^{-1}\| V_2(x_0)]^{\frac{1}{2}}.$$

Proof: Along every solution x: $[t_0,\infty) \to X$ with $x(t_0) = x_0$ the following holds a.e. :

$$\dot{V}_2(x(t)) < -\gamma\|P_2\bar{M}x(t)\|$$

$$< -\gamma[2\|P_2^{-1}\|^{-1} V_2(x(t))]^{\frac{1}{2}}$$

which, on integration, ensures that $V_2(x(t_f+t_0)) = 0$ $(\Leftrightarrow x(t_f+t_0) \in W)$ for some $t_f < \gamma^{-1}[2\|P_2^{-1}\| V_2(x_0)]^{\frac{1}{2}}$; moreover, $V_2(x(t)) = 0 \Leftrightarrow x(t) \in W$ $\forall\ t > t_f+t_0$, i.e. the subspace W is positively $\mathcal{H}_F$ -invariant.

□

Remark: Note that the upper bound on the time required to attain W is inversely proportional to the controller parameter $\gamma > 0$.

In view of the $\mathcal{H}_F$ -invariance of W, $\bar{M}x(\cdot) \equiv 0$ on every solution x: $[t_0,\infty) \to X$ of the feedback system with $x(t_0) = x_0 \in W$. Thus, motion in W is governed by

$$\left.\begin{aligned} &\dot{w}(t) \in L_1AM^* + L_1\mathcal{G}_1(t, M^*w(t)), \qquad w(t) \in R^{n-m} \\ &w(t_0) = L_1x_0 \end{aligned}\right\} \qquad (4.4.4)$$

in the sense that $x(\cdot)$: $[t_0,\infty) \to W$, $x(t_0) = x_0 \in W$, is a trajectory of the feedback controlled system (4.3.1) and (4.3.6)–(4.3.7) if and only if $x(\cdot) = M^* w(\cdot)$, where $w(\cdot)$: $[t_0,\infty) \to \mathbb{R}^{n-m}$ solves the initial value problem (4.4.4).

<u>Lemma 4.4.3</u>:Let $G_1 = \{0\}$. Then, for each $(t_0,x_0) \in \mathbb{R}\times W$, the feedback controlled differential inclusion, given by (4.3.1) and (4.3.6)–(4.3.7), admits a unique solution x: $[t_0,\infty) \to W$, with $x(t_0) = x_0$, given by

$$x(\cdot) = M^* \exp[L_1AM^*(\cdot - t_0)]L_1x_0 ,$$

where $\sigma(L_1AM^*) = \Lambda \subset \mathbb{C}^-$.

<u>Proof</u>:Immediately follows noting that, in the absence of residual uncertainty, i.e. for $G_1 = \{0\}$, (4.4.4) reduces to the ***asymptotically stable linear ideal model*** equation

$$\dot{w}(t) = L_1AM^*w(t), \qquad \sigma(L_1AM^*) = \Lambda \subset \mathbb{C}^-.$$

□

If residual uncertainty is present, i.e. if $G_1 \neq \{0\}$, then uniqueness of solutions and ideal model motion in W cannot be guaranteed. However, the feedback controlled system has an attractive compact set S (the diameter of which is proportional to the parameter κ_3), quantified in the following lemma and corollary.

<u>Lemma 4.4.4</u>:Define V_1: $\mathbb{R}^{n-m} \to \mathbb{R}^+$, $w \mapsto \frac{1}{2}\langle w, P_1w\rangle$. Then, if $\kappa_1 < \frac{1}{2}\|P_1\|^{-1}$,

$$\Sigma := \{w\colon V_1(w) \le \tfrac{1}{2}r^2\|P_1\|,\ r := 2\kappa_3[1-2\kappa_1\|P_1\|]^{-1}\|P_1\|\} \qquad (4.4.5)$$

is a globally uniformly asymptotically stable set for system (4.4.2).

Proof: Invoking A4.2, along every solution $w(\cdot)$ of (4.4.4) the following holds :

$$\dot{V}_1(w(t)) < -\tfrac{1}{2}[1-2\kappa_1\|P_1\|]\|w(t)\|^2 + \kappa_3\|P_1\|\|w(t)\| \qquad \text{a.e.}$$

from which the required result may be deduced by standard arguments.

□

Corollary 4.4.2: $S:= \{ x \in X: L_1 x \in \Sigma \} \cap W$ is a globally uniformly weakly attractive set for motion in W of the feedback controlled system (4.3.1) and (4.3.6)–(4.3.7).

Proof: An immediate consequence of the above lemma and the positive H_P – invariance of W.

□

Finally, for $(t_0,x_0) \in \mathbb{R}\times W$, let $\Delta x(\cdot)$ denote the deviation of a trajectory $x(\cdot)$ of the feedback system (with $x(t_0) = x_0$) from ideal model motion. Specifically,

$$\Delta x(t) = M^* y(t) \;, \qquad t > t_0,$$

where $$y(t) := L_1 x(t) - \exp[L_1 A M^*(t-t_0)] L_1 x_0 .$$

Lemma 4.4.5: If $\kappa_1 < \tfrac{1}{2}\|P_1\|^{-1}$, then

$$\|\Delta x(t)\| < 2\|M^*\|\|P_1\|[\|P_1^{-1}\|\|P_1\|]^{\frac{1}{2}}[\kappa_1\|P_1^{-\frac{1}{2}}\|\tilde{r}(L_1x_0)+\kappa_3] \qquad (4.4.6)$$

where

$$\tilde{r}(L_1x_0) := \begin{cases} \sqrt{2}V_1^{\frac{1}{2}}(L_1x_0) \;; & L_1x_0 \notin \Sigma \\ \\ r\|P_1\|^{\frac{1}{2}} \;; & L_1x_0 \in \Sigma. \end{cases}$$

Proof:Noting that $y(\cdot)$ solves the initial value problem

$$\dot{y}(t) \in L_1 AM^* y(t) + L_1 G_1(t,x(t)) , \quad y(t_0)=0,$$

and again invoking A4.2,

$$\dot{V}_1(y(t)) \le -\tfrac{1}{2}\|y(t)\|^2 +$$

$$\|P_1\|\|y(t)\|[\kappa_1\|L_1x(t)\| + \kappa_3] \quad \text{a.e..} \qquad (4.4.7)$$

Moreover, since $L_1x(\cdot)$ solves (4.4.2),

$$\dot{V}_1(L_1x(t)) \le -\tfrac{1}{2}[1 - 2\kappa_1\|P_1\|]\|L_1x(t)\|^2 +$$

$$\kappa_3\|P_1\|\|L_1x(t)\| \text{ a.e..} \qquad (4.4.8)$$

Combining (4.4.7) and (4.4.8), the bound on the deviation $\Delta x(\cdot)$ (given by (4.4.6)) is readily deduced.

□

There now follows a summary of the main results. For ***matched*** uncertainty, the following result is a direct consequence of lemmas 4.4.1–4.4.3.

Theorem 4.4.1:If $G_1 \equiv \{0\}$, then the generalized feedback F, given by (4.3.6)–(4.3.7), renders the zero state of the differential inclusion system (4.3.1) globally uniformly asymptotically stable. Moreover, the dynamic behaviour of the feedback controlled system ultimately corresponds to that of the ideal model in the sense that

$$x(t) = M^*w(t), \qquad \forall\ t \ge t_0 + \gamma^{-1}[\|P_2^{-1}\|\|P_2\|]^{\frac{1}{2}}\delta,$$

on every solution $x(\cdot)$ with $\|\bar{M}x(t_0)\| \le \delta$, where $w(\cdot)$ is a solution of the linear ideal model

$$\dot{w} = L_1 AM^* w$$

with prescribed spectrum $\sigma(L_1 AM^*) = \Lambda \subset \mathbb{C}^-$.

In the general case (i.e. in the presence of ***unmatched*** uncertainty), the following is a direct consequence of lemmas 4.4.1–4.4.2, 4.4.4 and corollary 4.4.2.

Theorem 4.4.2: If $\kappa_1 < \frac{1}{2}\|P_1\|^{-1}$, the generalized feedback F, given by (4.3.6)–(4.3.7), renders the compact set

$$S = \{\, x \in X \colon L_1 x \in \Sigma \,\} \cap W,$$

with Σ given by (4.4.5), globally uniformly asymptotically stable.

Remarks: (*i*) Note that, if the unmatched uncertainty parameter κ_3 is zero, then $S = \{0\}$.

(*ii*) The approach taken in this chapter generalizes to cases in which the nominal ***linear*** system is replaced by an asyptotically stable ***nonlinear*** system, provided that a Lyapunov function for this nominal system is available (see Corless and Leitmann [21]).

Consider an example which illustrates the preceding analysis. It is supposed that the controlled nonlinear uncertain system is :

$$\ddot{z}(t) + z(t) - \mu\dot{z}(t)[1 - \dot{z}^2(t)][4 - \dot{z}^2(t)] - [1 - b]u(t) = 0, \quad z(t), u(t) \in \mathbb{R}$$

where $0 < \mu < 1/3$ and $-\frac{1}{4} < b < \frac{1}{4}$ are unknown parameters. For this illustration and notational convenience, it is assumed that μ and b are constant. In particular, for $\mu = 1/3$, the phase portrait of the ***uncontrolled*** system is depicted in figure 8 (which shows an unstable equilibrium and two limit cycles – one stable and one unstable), shown below.

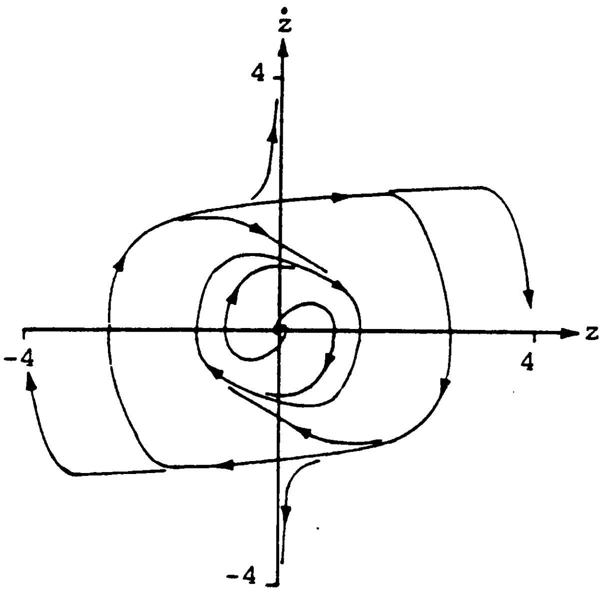

Figure 8: Phase portrait of the uncontrolled system (μ = 1/3).

Writing $x = [x_1 \quad x_2]^T = [z \quad \dot{z}]^T$, it is readily verified that A4.1 holds with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathcal{G}_1 \equiv \{0\},$$

$$\mathcal{G}_2(t,x) = \{\ \mu x_2(1-x_2^2)(4-x_2^2) \ :\ 0 < \mu < 1/3\ \} \subset \mathbb{R}, \quad \beta \equiv \tfrac{1}{4}.$$

In this case, all the uncertainty in the system is matched. Adopting the ideal model spectrum $\Lambda = \{-1\}$, the subspace W is given by

$$W = \{\ x\ :\ x_1 + x_2 = 0\ \}.$$

Selecting C = −1, the linear feedback operator F in (4.2.3) becomes F = [0 −2]. Solving (4.3.2) and (4.3.3), P_1 and P_2 are determined as $P_1 = \frac{1}{2} = P_2$. The associated multifunction $\mathcal{P}_2$ is

$$x \mapsto \mathcal{P}_2(x) = \begin{cases} [0,\infty)\ ; & x_1 + x_2 > 0 \\ \mathbb{R}\ ; & x_1 + x_2 = 0\ . \\ (-\infty,0]\ ; & x_1 + x_2 < 0 \end{cases}$$

Hence, the function $(t,x) \mapsto \xi((G_2(t,x) - ML_1 G_1(t,x)) \cap P_2(x))$, in (4.3.7), reduces to the function

$$x \mapsto \begin{cases} f^+(x_2) & ; \quad x_1 + x_2 > 0, \quad x_2 > 0 \\ |f(x_2)| & ; \quad x_1 + x_2 = 0 \\ |f^-(x_2)| & ; \quad x_1 + x_2 < 0, \quad x_2 < 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where f^+ and f^- denote the positive and negative parts of the function $f\colon x_2 \mapsto x_2(1-x_2^2)(4-x_2^2)/3$.

For a design parameter value $\gamma = 0.1$, figure 9 (shown below) depicts a typical family of trajectories of the feedback controlled system, wherein the attractivity of the subspace W is clearly evident.

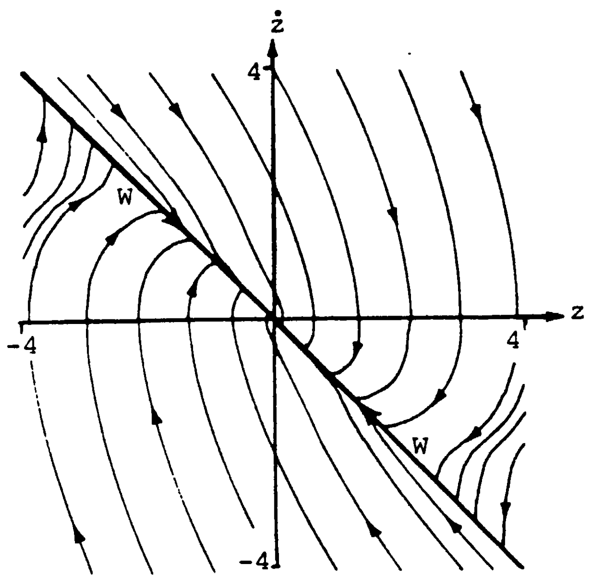

Figure 9: Invariant subspace W and typical trajectories of the feedback controlled system.

4.5 Robotics example revisited

In this chapter, feedback stabilization of controlled, uncertain dynamical systems has been investigated using a class of generalized feedbacks, of which selections are discontinuous in nature. In practice, this gives rise to "chattering" controls and hence, for practical considerations, it is desirable that the set Γ_F have non-zero Lebesgue measure. In order to overcome chattering controls, the switching manifold Γ_F can be replaced by an ε-region for which feedback controls that are continuous in state are used. This can be achieved by adopting a generalized feedback of the form :

$$(t,x) \mapsto F(t,x) = Fx - \rho(t,x) D_c(P_2\bar{M}x) \quad ,$$

where the multifunction D_c: U $\rightrightarrows$ U is defined by

$$u \mapsto D_c(u) := \begin{cases} \{\|u\|^{-1}u\}; & u \in U \backslash \varepsilon \bar{B}_U \\ \bar{B}_U \ ; & \text{otherwise.} \end{cases}$$

Remark:The manifold $\ker(P_2\bar{M}x)$ essentially plays the rôle of a switching surface as $\varepsilon \downarrow 0$.

In this case continuous selections of the multifunction D_c exist, as well as discontinuous ones, e.g. a particular continuous selection is

$$u \mapsto \begin{cases} \|u\|^{-1}u \ ; & u \in U \backslash \varepsilon \bar{B}_U \\ (\|u\| + \varepsilon)^{-1}u; & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is a design parameter. Of course, one could have used the approach taken in chapter 3, however this does not have the advantage of being able to prescribe stable linear dynamic behaviour on the manifold W.

For the example of the cylindrical robot, described in §3.6, the ensuing treatment adopts controls which are discontinuous in state.

Recall: A path, to be tracked by the robot arm, is defined in terms of the function $t \mapsto p(t)$. The error state, $t \mapsto e(t) := x(t) - p(t)$, which represents the deviation between the actual and reference trajectories, together with the deviation between corresponding velocity components, satisfies (see (3.6.4))

$$\left.\begin{aligned} &\dot{e}(t) \in Ae(t) + Bu(t) + B\mathcal{G}_p(e(t),u(t)), \\ &e(t_0) = e_0 = x_0 - p(t_0), \end{aligned}\right\} \qquad (4.5.1)$$

where A, B are defined in (3.6.2) and $\mathcal{G}_p$ is given by (3.6.6).

By definitions of A, B and $\mathcal{G}_p$, hypothesis A4.1 is satisfied with $\mathcal{G}_1 \equiv \{0\}$, $\mathcal{G}_2(t,e) = \mathcal{E}(e+p(t)) - p^*(t)$, and $\beta(t) = \kappa < 1$, for all $t \in [t_0,\infty)$, where $\mathcal{E}$, p^* and κ are given by (3.6.3), (3.6.5), (3.6.7), respectively.

The notation O_{nxn} and I_{nxn} is introduced for the zero and identity matrices defined on $\mathbb{R}^{nxn}$.

Let $L_1 = [I_{3x3} \quad O_{3x3}]$, then, since $L_2 = [O_{3x3} \quad I_{3x3}]$, $L = I_{6x6} = L^{-1}$ which implies $R_1 = [I_{3x3} \quad O_{3x3}]^T$.

Selecting $\Lambda = \{ \lambda,\mu,\nu \} \subset \mathbb{C}^-$, M is determined such that $\sigma(L_1A(R_1+BM)) = \Lambda$.

One such M is

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}.$$

The linear component F of overall control design is now given by $F = C\bar{M} - \bar{M}A$, where $\bar{M} = L_2 - ML_1 = [-M \quad I_{3x3}]$ and $\sigma(C) \subset \mathbb{C}^-$. On the manifold W, C governs how quickly the state origin is reached. Therefore C is chosen (for convenience) to be a scalar multiple of M, where the constant scalar is selected so that there is good convergence to the state origin, i.e. $C = \alpha M$, with $\alpha > 1$. Thence, $F = [-\alpha M^2 \quad (\alpha+1)M]$.

The Lyapunov equation (4.3.3) is solved to give $P_2 = -(2\alpha M)^{-1}$.

Thus the nonlinear component of the control is $-\rho(t,e)\mathcal{D}(P_2\bar{M}e)$, where $\rho(t,e)$ satisfies

$$\rho(t,e) > (1-\kappa)^{-1}[\kappa\|Fe\| + \xi(\mathcal{G}_2(t,e) \cap \mathcal{P}_2(e)) + \gamma],$$

the multifunction $\mathcal{P}_2(e)$ is defined in (4.3.5), γ is a design parameter, $P_2\bar{M} = -(2\alpha)^{-1}[-I_{3x3} \quad M^{-1}]$, and

$$\mathcal{D}(u) = \begin{cases} \{\|u\|^{-1}u\}; & u \neq 0 \\ \bar{\mathcal{B}}_{R^3}\ ; & u = 0. \end{cases}$$

A specific example of a discontinuous feedback control is

$$u(t) = [-\alpha M^2 \quad (\alpha+1)M]e(t) - \rho(t,e)\begin{cases} \|P_2\bar{M}e\|^{-1}P_2\bar{M}e, & P_2\bar{M}e \neq 0 \\ v, & P_2\bar{M}e = 0 \end{cases} \tag{4.5.2}$$

where $P_2\bar{M}e = \frac{1}{2}\alpha^{-1}[e_1-\lambda^{-1}e_4 \quad e_2-\mu^{-1}e_5 \quad e_3-\nu^{-1}e_6]^T$, $v \in R^3$ satisfies $\|v\| < 1$, and a suitable choice for $\rho(t,e)$ is

$$\rho(t,e) = (1-\kappa)^{-1}\{\ \kappa\|[-\alpha M^2 \quad (\alpha+1)M]e\| + g(t,e)\ \}$$

with $(t,e) \mapsto g(t,e)$ defined in (3.6.9). Hence, invoking theorem 4.4.1, the discontinuous feedback control u, given above (see (4.5.2)), renders the zero state of (4.5.1) globally uniformly asymptotically stable. Ultimately, for all

$$t > t_0 + \gamma^{-1}[\min\{\lambda,\mu,\nu\}\max\{\lambda,\mu,\nu\}]\delta,$$

the dynamic behaviour of the feedback controlled system is given by

$e(t) = [I_{3x3} \quad M]^T w(t)$ on every solution $e(\cdot)$ with $||\bar{M}e(t_0)|| < \delta$, where $w(\cdot)$ is a solution of the linear model :

$$\dot{w}(t) = Mw(t) \qquad \text{with } \sigma(M) = \Lambda.$$

5.FEEDBACK STABILIZATION OF A CLASS OF NONLINEARLY COUPLED UNCERTAIN DYNAMICAL SYSTEMS

5.1 Introduction

The theory developed in chapter 4 relates to asymptotic feedback stabilization of an uncertain system by construction of a linear manifold and a discontinuous control such that the manifold is globally attractive and, if the uncertainty is matched, on which all motion is independent of the uncertainty in the system. In this chapter, the above work is adapted to the problem of stabilizing, by feedback, a system which consists of two bilinearly coupled subsystems, together with uncertainty in each of the subsystems (see Goodall and Ryan [30]). A class of discontinuous feedback controls is presented which guarantees global asymptotic stability of the zero state of the pair of nonlinearly coupled uncertain dynamical systems. For this problem, it transpires that the manifold is, in fact, nonlinear and smooth. As in chapters 3 and 4, a formulation based on differential inclusions is adopted wherein system uncertainty is modelled by set-valued maps.

The prototype control system to be considered has the structure of two bilinearly coupled subsystems and has a non-asymptotically-stabilizable linearization. This system is assumed to be subject to uncertainty, modelled by additional nonlinearities in the coupling terms and by augmenting the nominal differential equations by a set-valued map, thereby giving rise to a controlled differential inclusion. Within this framework, the problem of stabilization is considered and a class of discontinuous feedback controls is developed which renders the system zero state globally attractive.

The approach taken here is to break down the problem into two separate stages. In the first stage a nonlinear manifold W is constructed with the property that, if it is rendered invariant by suitable feedback, then all solutions in W tend to the state origin. The results of Slemrod [75] on bilinear stabilization are invoked at this stage. In the second stage a feedback control is designed which renders (*i*) W both invariant and globally finite-time attractive, (*ii*) the zero state of the differential inclusion system globally attractive. In designing the feedback control law, the procedure follows closely that used by Goodall and Ryan [28].

Stability of systems, similar in nature to the prototype, have also been investigated by Aeyels [1], Brockett [13], and Carr [14]. In the context of their work, the manifold W can be interpreted as a global ***centre manifold.***

The class of nonlinearly coupled uncertain dynamical systems is described in §5.2 and the problem definition is given in §5.3. The outline of the method is given in §5.4 and the class of generalized stabilizing feedback controls is described in §5.6. With these generalized feedbacks, the manifold W is shown to be invariant and attractive and the stability of the system is guaranteed (see §5.7). Finally, in §5.8, the theory is illustrated by investigating an example of a controlled bilinear system which models a deformable column by a double pendulum connected by elastic hinges (see Slemrod [75]).

5.2 The class of nonlinearly coupled uncertain dynamical systems

For the whole of this chapter, X, Y denote the sets $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, which are the state spaces for the two subsystems, and U denotes the set $\mathbb{R}^q$, the control space for the system. The prototype control system (with control input u(t)) consists of two bilinearly coupled subsystems :

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{q} y_i(t)\ B_i x(t)\ , \quad x(t) \in X,\ y(t) \in Y, \tag{5.2.1a}$$

$$\dot{y}(t) = A_1 y(t) + u(t)\ , \quad u(t) \in U, \tag{5.2.1b}$$

where $y(t)=[y_1(t),\ldots,y_q(t)]^T$, A_0, $B_i \in R^{p\times p}$ and $A_1 \in R^{q\times q}$ are known matrices defining the nominal bilinearly coupled system. Attention is restricted to the subclass of systems (5.2.1a) - (5.2.1b) for which the following hypothesis holds :

A5.1: There exists a real symmetric matrix $K > 0$ such that

$$KA_0 + A_0^TK + J = 0,$$

where $J \geq 0$ is a real symmetric matrix.

The most interesting problem (i.e. the critical case) occurs when $J = O$, for in this case the hypothesis implies that the eigenvalues of A_0 are either zero or purely imaginary (see proposition 1.5.9) and A_0 is stable, but not asymptotically stable. Hence the nominal system has a stabilizable, but not necessarily asymptotically stabilizable, linearization.

Consider now the above system subject to uncertainty. It is supposed that the uncertainty can be modelled by a nonlinear perturbation to equation (5.2.1). Specifically, the controlled uncertain dynamical system is of the form :

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{q} (y_i(t) + f_i(x(t),y(t)))B_ix(t), \quad x(t)\in X, \; y(t)\in Y \qquad (5.2.2a)$$

$$\dot{y}(t) \in A_1y(t) + u(t) + G(x(t),y(t),u(t)), \quad u(t) \in U, \qquad (5.2.2b)$$

where f_i: $X\times Y \to R$ are unknown, real-valued functions and G: $X\times Y\times U \rightrightarrows U$ is a known multifunction (set-valued map), encompassing all possible perturbations to the nominal differential equations, modelling the uncertainty in the system.

5.3 Problem statement

Defining a class C of ***generalized feedbacks*** in a similar way to those generalized feedbacks that are used in chapters 3 and 4 (see definition 3.4.1), the objective may now be loosely stated as that of determining a generalized feedback law $u(t) \in F(x(t),y(t))$ which renders the zero state of the feedback controlled differential inclusion system :

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{q} (y_i(t) + f_i(x(t),y(t)))B_ix(t) \tag{5.3.1a}$$

$$\dot{y}(t) \in A_1y(t) + F(x(t),y(t)) + G(x(t),y(t),F(x(t),y(t))), \tag{5.3.1b}$$

where $G(x,y,F(x,y)) := \bigcup_{u \in F(x,y)} G(x,y,u)$,

globally uniformly asymptotically stable in the sense of definition 2.2.8, i.e. the feedback controlled differential inclusion system (5.3.1) is ***globally uniformly asymptotically stable.***

5.4.Outline of method

For this problem, the method for determining a generalized feedback control law is broken down into two stages.

Stage I

In the first stage, subsystem (5.3.1a) is regarded as an isolated system with input y and a smooth feedback function $h: x \mapsto y = h(x)$, $X \to Y$, is sought to stabilize this system. This is achieved, under the following hypotheses, by an approach akin to that of Slemrod [75].

A5.2:(a)For each i, f_i is continuous and $f_i(x,0) = 0 \quad \forall x$;

(b)There exist known real constants $\beta \in [0,1)$, $\delta > 0$ and known function $\varphi: \mathbb{R} \to \mathbb{R}$ satisfying

(i) $\varphi \in C^1(\mathbb{R})$ and (ii) $\delta v^2 < v\varphi(v), \quad \forall v \in \mathbb{R}$,

such that, for all i,

$|f_i(x,y)| < |\varphi(\langle x, KB_i x\rangle)| + \beta |y_i|$;

(c)There exists a nonempty set $\Omega \subset X \backslash \{0\}$ such that

$(i)$for each $x \in \Omega$ there exists a positive integer k such that

$\text{span}\{Jx,\ A_0 x,\ ad^0(A_0,B_i)x,\ ad^1(A_0,B_i)x,\ldots,ad^k(A_0,B_i)x\ ;\ i=1,..,q\} = X$

and (ii) $\{0\}$ is the only subset of $\Omega^c \cap \Gamma$ which is invariant under $\exp(A_0 t)$, where Ω^c denotes the complement of Ω and

$\Gamma := \text{Ker}(J^{\frac{1}{2}}) \cap \{\ x \in X: \langle x, KB_i x\rangle = 0,\ i= 1,..,q\ \}$.

Remark:Conditions (a) and (b) are strong structural conditions on the uncertainties f_i; condition (c) is essentially that of Ryan and Buckingham [66], a weaker version of that of Slemrod [75]. The operators $ad^j(A_0,B_i)$ are defined recursively as follows:

$ad^0(A_0,B_i) := B_i; \qquad ad^1(A_0,B_i) \equiv ad(A_0,B_i) := A_0 B_i - B_i A_0,$

$ad^j(A_0,B_i) := ad(A_0, ad^{j-1}(A_0,B_i)),\ j= 2,3,\ldots.$

Stage II

For the overall system, the relation $y = h(x)$ (with h prescribed in §5.5) defines a smooth manifold W in the state space X×Y. The second stage of the method is to determine a generalized feedback law $(x,y) \mapsto F(x,y)$ which renders the state origin globally uniformly asymptotically stable. More specifically, F is designed so as to render the manifold W globally uniformly finite-time attractive (see definition 2.2.10) and invariant. Thus, all solutions of the feedback controlled system ultimately attain the manifold W; motion thereafter is constrained to W and hence (by the analysis of stage I) tends asymptotically to the origin. This feedback design is achieved, under the following additional hypothesis (a structural condition on G), by a procedure that follows closely that used by Goodall and Ryan [28].

A5.3: There exists a known multifunction G_s: X×Y $\rightrightarrows$ U, which is upper semicontinuous with convex and compact values, and a real constant $\kappa \in [0,1)$ such that

$$G(x,y,u) = G_s(x,y) + \kappa\|u\|\bar{B}_U .$$

5.5. Determination of the manifold W

In this section, a feedback function $x \mapsto y(x) \in Y$ is constructed which renders the zero state of system (5.3.1a) globally uniformly asymptotically stable. The approach is akin to that used by Slemrod [75] which uses Lyapunov theory and the ***invariance principle*** of LaSalle. This feedback control law is then used to define the manifold W.

Lemma 5.5.1: Under the conditions stated in the hypotheses A5.1 and A5.2, the feedback function $x \mapsto y = [y_1, y_2, \ldots, y_q]^T$, where

$$y_i := -\alpha(1-\beta)^{-1}\varphi(\langle x, KB_i x\rangle) \quad \text{and } \alpha > 1,$$

renders the zero state of system (5.3.1a) globally uniformly asymptotically stable.

Proof: The hypotheses A5.2(a) and (b) ensure that, for each $x^0 \in$ X, there exists at least one (maximal) solution $x(\cdot)$: $[0,\tau) \to$ X with $x(0) = x^0$ of feedback controlled system (5.3.1a). The behaviour of the function V_1: $R^p \to R_0^+$ along solutions is examined, where V_1 is defined by $V_1(x) := \frac{1}{2}\langle x, Kx\rangle$. Under the conditions stated in hypothesis A5.1 and A5.2(b), a straightforward calculation shows that, along every maximal solution of (5.3.1a), the following holds for almost all $t \in [0,\tau)$:

$$\dot{V}_1(x(t)) \leq -\tfrac{1}{2}\|J^{\frac{1}{2}}x\|^2 - (\alpha-1)\sum_{i=1}^{q} |\varphi(\langle x(t), KB_i x(t)\rangle)|\ |\langle x(t), KB_i x(t)\rangle| \leq 0,$$

which implies that all solutions can be continued indefinitely. Also, properties of boundedness of solutions and stability clearly hold. Applying the invariance principle of LaSalle (see LaSalle [45]), it may be concluded that all solutions of (5.3.1a) approach (as $t \to \infty$) a set Λ, which is the largest invariant subset of Γ (defined in A5.2(c)). For system (5.3.1a), with $x^0 \in \Lambda$, the state $x(t)$ satisfies: $J^{\frac{1}{2}}x(t) = 0$, $\langle x(t), KB_i x(t)\rangle = 0$ for all i, and $\dot{x}(t) = A_0 x(t)$ for all $t \in \mathbb{R}$. Slemrod [75] has shown that this characterizes the set Λ in the form

$$\Lambda := \{\ x \in X:\ J^{\frac{1}{2}}x = 0,\ \langle x, KA_0 x\rangle = 0,\ \langle x, K\,\mathrm{ad}^j(A_0,B_i)x\rangle = 0;\ j \in Z_0^+,\ i = 1,..q\ \}.$$

As a consequence of A5.2(c), following an argument applied by Ryan and Buckingham [66], one may deduce that $\Lambda = \{0\}$ and hence global uniform asymptotic stability of the origin may be concluded.

□

Introducing the function $x \mapsto h(x) = [h_1(x), h_2(x),..., h_q(x)]^T$, where $h_i(x) := -\alpha(1-\beta)^{-1}\varphi(\langle x, KB_i x\rangle)$ and $\alpha > 1$, the manifold $W \subset X\times Y$ is defined as

$$W := \{\ y \in Y:\ y = h(x)\ \}.$$

5.6. The class of generalized feedback controls $\mathcal{C}$

It may be assumed (w.l.o.g.) that the eigenvalues of A_1 lie in the open left half of the complex plane. Let $P > 0$ be the unique symmetric solution of the Lyapunov equation

$$PA_1 + A_1^T P + Q = 0 \tag{5.6.1}$$

where $Q > 0$ is a design parameter. The multifunctions $\mathcal{P}$: $X \times Y \rightrightarrows U$ and $\mathcal{D}$: $U \rightrightarrows U$ are defined by

$$(x,y) \mapsto \mathcal{P}(x,y) := \{ \nu \in U: \langle \nu, Py \rangle \geq \langle \nu, Ph(x) \rangle \}$$

$$u \mapsto \mathcal{D}(u) := \begin{cases} \{\|u\|^{-1}u\}, & u \neq 0 \\ \bar{B}_U, & u = 0. \end{cases}$$

The proposed feedback, for stabilization of the system (5.3.1), is taken to be the multifunction

$$(x,y) \mapsto \mathcal{F}(x,y) := -A_1 h(x) + Dh(x)\{ A_0 x + \sum_{i=1}^{q} y_i B_i x \}$$
$$- \rho(x,y)\mathcal{D}(P(y-h(x))), \tag{5.6.2}$$

where $Dh(x) \in \mathbb{R}^{q \times p}$ is the Fréchet derivative of h: $X \to Y$ at x, ρ is any continuous functional satisfying

$$\rho(x,y) \geq \rho_0(x,y) := (1-\kappa)^{-1}\{ \kappa\|-A_1 h(x) + Dh(x)\{ A_0 x + \sum_{i=1}^{q} y_i B_i x \}\|$$
$$+ \sum_{i=1}^{q} (\beta|y_i| + \alpha^{-1}(1-\beta)|h_i(x)|)\|Dh(x)B_i x\|$$
$$+ \xi(\mathcal{G}_s(x,y) \cap \mathcal{P}(x,y)) + \gamma \}, \tag{5.6.3}$$

and $\gamma > 0$ is a design parameter. Note that the continuity of ρ and the upper semicontinuity of D ensure that F is upper semicontinuous, and clearly takes convex and compact values; also, F is singleton-valued except on a set Γ_F. Hence F qualifies as a generalized feedback.

Remarks:(*i*) If A_1 is not asymptotically stable append the generalized feedback F with a term $(A^*-A_1)(y-h(x))$, where A^* is an asymptotically stable matrix, and replace A_1 in (5.6.1) by A^*.

(*ii*)The intersection $G_s(x,y) \cap P(x,y)$ is adopted in (5.6.3) in order to economize on the gain ρ by exploiting the possible occurrence of "stability enhancing" uncertainties.

5.7. Asymptotic stability of the feedback controlled differential inclusion system

Defining $e(t) := y(t) - (h \circ x)(t)$, the function $t \mapsto e(t) \in Y$ satisfies the differential inclusion

$$\dot{e}(t) \in A_1 e(t) + H(x(t), e(t), F(x(t), e(t)+h(x(t)))) \qquad (5.7.1)$$

where

$$H(x,e,F(x,e+h(x))) := G(x,e+h(x),F(x,e+h(x))) - \rho(x,e+h(x))D(Pe) - Dh(x)\sum_{i=1}^{q} f_i(x,e+h(x))B_i x$$

and F is given by (5.6.2). It can be readily verified that the multifunction G is upper semicontinuous with convex and compact values.

Consider the system :

$$\begin{bmatrix} \dot{x}(t) \\ \\ \dot{e}(t) \end{bmatrix} \in \tilde{G}(x(t), e(t)) \qquad (5.7.2a)$$

with initial condition

$$\begin{bmatrix} x(0) \\ \\ e(0) \end{bmatrix} = \begin{bmatrix} x^0 \\ \\ e^0 \end{bmatrix}, \qquad (5.7.2b)$$

where the multifunction $\tilde{G}$: $X \times Y \rightrightarrows R^n$, $n = p + q$, is defined by

$$\tilde{G}(x,e) := \left\{ \begin{bmatrix} A_0 x + \sum_{i=1}^{q} (e_i + h_i(x) + f_i(x, e+h(x)))B_i x \\ \\ A_1 e + \eta \end{bmatrix} : \eta \in H(x, e, F(x, e+h(x))) \right\}. \qquad (5.7.2c)$$

Clearly the multifunction $(x,e) \mapsto \tilde{G}(x,e)$ is upper semicontinuous with convex and compact values.

Hence, for each pair $\begin{bmatrix} x^0 \\ e^0 \end{bmatrix}$, the initial value problem (5.7.2) admits a maximal solution $\begin{bmatrix} x \\ e \end{bmatrix} : [0,\tau) \to R^n$ (see propositions 3.2.1 and 3.2.2).

Define V_2: $R^n \to R_0^+$, a Lyapunov function candidate, as

$$V_2(\begin{bmatrix} x \\ \\ e \end{bmatrix}) := \tfrac{1}{2} \langle \begin{bmatrix} x \\ \\ e \end{bmatrix}, L \begin{bmatrix} x \\ \\ e \end{bmatrix} \rangle, \qquad L = \begin{bmatrix} K & 0 \\ \\ 0 & \zeta P \end{bmatrix},$$

where ζ is a real positive constant to be specified.

Along each maximal solution $\begin{bmatrix} x \\ e \end{bmatrix}$: $[0,\tau) \to R^n$,

$$\dot{V}_2(\begin{bmatrix} x(t) \\ \\ e(t) \end{bmatrix}) \in L(x(t), e(t)) := L_1(x(t), e(t)) + L_2(x(t), e(t)), \text{ where}$$

$$L_1(x,e) := \{ \langle A_0 x + \sum_{i=1}^{q} (e_i + h_i(x) + f_i(x, e+h(x)))B_i x, \; Kx \rangle \} \subset R \quad \text{and}$$

$L_2(x,e) := \zeta\langle\langle A_1 e + H(x,e,F(x,e+h(x))),\ Pe\rangle\rangle \subset \mathbb{R}.$

Thus it is required to show that $L(x,e) \cap [0,\infty) = \emptyset$, for all $(x,e) \in X\times Y$.

It is easily shown that

$$\max L_1(x,e) \le -\tfrac{1}{2}\|J^{\frac{1}{2}}x\|^2 + \sum_{i=1}^{q}\{-\delta(\alpha-1)|\langle x,\ KB_i x\rangle|^2 + (1+\beta)|e_i|\ |\langle x,\ KB_i x\rangle|\}.$$

Using (5.6.1) and the hypotheses A5.2 – A5.3,

$$\max L_2(x,e) \le \zeta[-\tfrac{1}{2}\langle e,\ Qe\rangle + \kappa\xi(F(x,e+h(x)))\|Pe\| +$$

$$\langle\langle G_s(x,e+h(x)) - \rho(x,e+h(x))D(Pe) - Dh(x)\sum_{i=1}^{q} f_i(x,e+h(x))B_i x,\ Pe\rangle\rangle],$$

whence, in view of (5.6.2) and (5.6.3),

$$\max L_2(x,e) \le -\tfrac{1}{2}\zeta(\langle e,\ Qe\rangle + 2\gamma\|Pe\|) \tag{5.7.3}$$

$$\le -\tfrac{1}{2}\zeta\lambda\sum_{i=1}^{q}|e_i|^2,$$

where $\lambda = \sigma_{\min}(Q)$. Hence

$$\max L(x,e) \le -\tfrac{1}{2}\|J^{\frac{1}{2}}x\|^2 - \tfrac{1}{2}\sum_{i=1}^{q}\left\langle\begin{bmatrix}|\langle x,\ KB_i x\rangle|\\ |e_i|\end{bmatrix}, E_\zeta\begin{bmatrix}|\langle x,\ KB_i x\rangle|\\ |e_i|\end{bmatrix}\right\rangle$$

where $E_\zeta := \begin{bmatrix}2\delta(\alpha-1) & -(1+\beta)\\ -(1+\beta) & \zeta\lambda\end{bmatrix}.$

Choosing $\zeta = (1+\beta)^2[2\delta(\alpha-1)\lambda]^{-1}$ ensures that, along all maximal solutions of (5.7.2),

$$\dot{V}_2\left(\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}\right) < 0, \quad \text{a.e..}$$

Therefore every maximal solution $\begin{bmatrix} x \\ e \end{bmatrix}: [0,\tau) \to R^n$ can be continued indefinitely and the properties of boundedness of solutions and stability readily follow.

It is now shown that the manifold W is finite-time attractive and invariant. Consider the behaviour of the function V_3: $Y \to R_0^+$, $e \mapsto V_3(e) := \frac{1}{2}\langle e, Pe\rangle$ along solutions of (5.7.1).

Lemma 5.7.1:For each $x^0 \in X$, the manifold W is attained in finite time t_f, satisfying

$$t_f < \gamma^{-1}\{2\|P^{-1}\|V_3(e^0)\}^{\frac{1}{2}},$$

and $x(t) \in W$ for all $t > t_f$.

Proof:From (5.7.3) it follows that, for $Pe \neq 0$,

$$\begin{aligned}\dot{V}_3(e(t)) &< -\gamma\|Pe(t)\| \\ &< -\gamma\sqrt{2}\|P^{-1}\|^{-\frac{1}{2}}\{V_3(e(t))\}^{\frac{1}{2}},\end{aligned}$$

a.e. along all solutions of (5.7.1).

Integration shows that the time, t_f, taken to attain W satisfies

$$t_f < \gamma^{-1}\{2\|P^{-1}\|V_3(e^0)\}^{\frac{1}{2}}$$

(which is inversely proportional to the design parameter γ). Moreover, $V_3(e(t)) = 0$ for all $t > t_f$ and hence $x(t) \in W$ for all $t > t_f$.

□

On W, motion is as earlier analysed in lemma 5.5.1.

Remark:The manifold W is in fact smooth and qualifies as a ***centre manifold*** for system (5.3.1) (see Carr [14]).

As a consequence of the above analysis, the following theorem can be concluded.

Theorem 5.7.1:The generalized feedback $F \in C$, given by (5.6.2), renders the zero state of the differential inclusion system (5.3.1) globally uniformly asymptotically stable.

5.8 Illustrative example

One example of a controlled bilinear system in elasticity is a system which models a deformable column by a double pendulum connected by elastic hinges (see Slemrod [75]). The linearized equation of motion for such a system has the form :

$$\ddot{y}(t) + Sy(t) + z(t)Hy(t) = 0 , \qquad (5.8.1)$$

where $S \in \mathbb{R}^{n\times n}$ is a diagonalizable matrix with positive eigenvalues, $H \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^n$, and $z \in \mathbb{R}$ is the control. For this model, it is not unrealistic to hypothesize that the dynamics of the actuator, governing the control, cannot be precisely defined. For example, suppose the dynamics can be modelled by the differential inclusion system :

$$\dot{z}(t) \in az(t) + (1+\psi)u(t) + \mathcal{H}(y(t),\dot{y}(t),z(t)) \qquad (5.8.2)$$

where $a < 0$ is a known constant, ψ is an unknown parameter satisfying $|\psi| < \kappa$ such that $\kappa \in (0,1)$ is known, u is the input to the actuator system, and $\mathcal{H}$: $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a known upper semicontinuous, multifunction with convex and compact values. The system, given in (5.8.1), has state space form :

$$\dot{x}(t) = A_0 x(t) + z(t) B x(t) , \tag{5.8.3}$$

where

$$x = \begin{bmatrix} y \\ \\ \dot{y} \end{bmatrix}, \quad A_0 = \begin{bmatrix} O & I \\ \\ -S & O \end{bmatrix}, \quad B = \begin{bmatrix} O & O \\ \\ -H & O \end{bmatrix} .$$

Defining $(x,z) \mapsto Z(x,z) := H(y,\dot{y},z)$, the differential inclusion system (5.8.2) can be written as

$$\dot{z}(t) \in az(t) + (1+\psi)u(t) + Z(x(t),z(t)) . \tag{5.8.4}$$

In particular, consider the matrices A_0 and B having the same form as those specified in one of the examples formulated by Slemrod [75], *viz.*

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & \frac{1}{2} & 0 & 0 \\ 2 & -3/2 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} .$$

Assumption A5.1 is satisfied with J = O and

$$K = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3/2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Consider hypothesis A5.2(c).

For $x \in R^4$, $\det\{ A_0 x,\ Bx,\ ad^1(A_0,B)x,\ ad^2(A_0,B)x \} = -\frac{1}{2} x_2^3 x_3$.

Defining $\Omega := \{ x \in R^4 : x_2 \neq 0,\ x_3 \neq 0 \}$ it follows that

$$\mathrm{span}\{ A_0 x,\ Bx,\ ad^1(A_0,B)x,\ ad^2(A_0,B)x \} = R^4, \quad \forall\, x \in \Omega.$$

Moreover, since $\langle x, KBx \rangle = -x_2 x_4$,

$$\Omega^c \cap \Gamma = \{x \in R^4 : x_2 = 0,\ x_3 = 0 \} \cap \{x \in R^4 : x_2 = 0,\ x_4 = 0\}$$

and $\{0\}$ is the largest subset invariant under $\exp(A_0 t)$, $t \in R$. In order to confirm A5.2(b), φ is chosen to be $\varphi(v) = v$, where $v = \langle x, KBx\rangle = -x_2 x_4$. The (centre) manifold for system (5.8.3)–(5.8.4) is given by

$$M = \{ z \in R: z = \alpha(1-\beta)^{-1} x_2 x_4 \ ; \ \alpha > 1,\ \beta \in [0,1);\ x_2,\ x_4 \in R \}.$$

Selecting $Q = 1$, the solution to the Lyapunov equation (5.6.1) is $P = -1/(2a)$. Taking $h(x) := \alpha(1-\beta)^{-1} x_2 x_4$, the multifunction P is

$$(x,z) \mapsto P(x,z) = \begin{cases} R_0^+ \ ; & z > h(x) \\ R \ ; & z = h(x) \ . \\ R_0^- \ ; & z < h(x) \end{cases}$$

The stabilizing feedback is given by

$$F(x,z) = -ah(x) + \alpha(1-\beta)^{-1}[x_4^2 + x_2(2x_1 - x_2(z+3/2))]$$
$$- \rho(x,z) D(z-h(x)),$$

where ρ satisfies

$$\rho(x,z) > (1-\kappa)^{-1}\{\kappa | -ah(x) + \alpha(1-\beta)^{-1}[x_4^2 + x_2(2x_1 - x_2(z+3/2))]|$$
$$+ (\alpha\beta(1-\beta)^{-1}|z| + |h(x)|)x_2^2 + \xi(Z(x,z) \cap P(x,z)) + \gamma \}, \quad (5.8.4)$$

$$\text{and} \quad D(z-h(x)) = \begin{cases} \operatorname{sgn}(z-h(x)), & z \neq h(x) \\ [-1,1] \ , & z = h(x). \end{cases}$$

For example, if $Z(x,z) := \{ \mu k(x,z): \mu \in [s_1, s_2],\ s_2 > s_1 > 0 \} \subset R$, where s_1, s_2 and the function $k: R^4 \times R \to R$ are known but the parameter μ is unknown, then the function $(x,z) \mapsto \xi(Z(x,z) \cap P(x,z))$, appearing in the inequality (5.8.4), reduces to the function

$$x \mapsto \begin{cases} s_2 k^+(x,z) \; ; & z > h(x), \quad k(x,z) > 0 \\ s_2 |k(x,z)| \; ; & z = h(x) \\ s_2 k^-(x,z) \; ; & z < h(x), \quad k(x,z) < 0 \\ \quad 0 \quad ; & \text{otherwise}, \end{cases}$$

where k^+ and k^- denote the positive and negative parts of the function $(x,z) \mapsto k(x,z)$.

6.CONCLUDING REMARKS AND SUGGESTIONS FOR FUTURE RESEARCH

6.1 Concluding remarks

In this thesis, a deterministic approach to the design of stabilizing feedback controls for classes of uncertain dynamical systems has been considered. Systems modelled by both differential equations and differential inclusions have been investigated. For a certain type of structured uncertainty, relating to functional properties and bounds, classes of nonlinear feedback controls, both continuous and discontinuous, have been designed which guarantee global uniform asymptotic stability of a compact set, containing the state origin. Both matched and unmatched structured uncertainty have also been considered. Moreover, under a matched uncertainty hypothesis, a class of generalized feedback controls, with practical analogues in the form of discontinuous feedbacks, has been presented which guarantees global uniform asymptotic stability of the zero state and ultimate attainment of prescribed ideal model behaviour. These basic stability problems have also been extended to include problems of tracking and model-following. In particular, to illustrate the tracking problem, an application from robotics has been investigated.

Finally, this work has been adapted to a particular problem of stabilizing, by feedback, a class of nonlinearly coupled uncertain dynamical systems, in which system uncertainty is modelled by set-valued maps. The uncertain dynamical systems are based on a prototype system that has the structure of two bilinearly coupled subsystems and has a non-asymptotically-stabilizable linearization.

6.2 Areas for further research

The work in this thesis can be developed in a number of different areas.

For example, the development of this work may be continued through investigation of some or all of the following proposals:

(a)*Variation of hypotheses for differential inclusion systems*

The theory developed in this thesis relating to feedback controlled differential inclusion systems assumed that the appropriate multifunction is upper semicontinuous, with compact and convex values. These hypotheses are sufficient for existence of local solutions. For a particular control problem, it may be such that one (or more) of these hypotheses is not appropriate. Thus a study of alternative hypotheses and investigation of their consequences, with respect to stability of the feedback control system, is warranted.

(b)*Perturbation about nominal nonlinear systems*

The basic structure, modelling the uncertainty in the control systems (considered in this thesis), consists of a known, linear, nominal system, together with a nonlinear perturbation of the nominal system. It is possible that this restriction, imposed by the structure, could be relaxed so that the uncertainty is modelled as a nonlinear perturbation about a known, nonlinear, nominal system. It is implicitly assumed that a Lyapunov function is available for the nonlinear, nominal system.

(c)*Nonlinear attractive manifolds*

In chapter 4, two deterministic theories : Lyapunov–based theory and Variable Structure Systems theory are combined in a unifying approach. The basis of the Variable Structure Systems theory is an attractive linear manifold (of the underlying state space), with an associated linear ideal model. The work in chapter 4 may generalize to encompass investigations of attractive nonlinear manifolds, with associated nonlinear ideal models.

(d)*Robust stability*

Robust stability of linear systems with structured perturbations has been investigated by, for example, Hinrichsen and Pritchard [38] and Pritchard and Townley [60], utilizing the concept of stability radius. The ideal model :

$$\dot{w}(t) = L_1A(R_1 + BM)w(t) , \qquad w(t) \in \mathbb{R}^{n-m},$$

introduced in chapter 4, involves a parameter M. It is proposed that the stability radius of $L_1A(R_1+BM)$ be maximized over the set of matrices $M \in \mathbb{R}^{m\times(n-m)}$ (see chapter 4). The matrix M assigns the spectrum of the linear ideal model which determines implicitly the attractive linear manifold arising from the Variable Structure Systems theory used in chapter 4.

(e)*Adaptive output feedback laws*

A further proposal is that the work on controlled differential inclusion systems be extended to problems of adaptive stabilization. Adaptive modifications to unknown system parameters, using generalized feedbacks, can be studied for the basic problem of stabilization by feedback.

(f)*Singularly perturbed uncertain dynamical systems*

An investigation into a class of uncertain dynamical systems which can be decomposed into two coupled subsystems (one, or more, of which consists of a controlled differential inclusion subsystem) by means of a singular perturbation parameter is envisaged. This investigation could consider the robustness of some desired stability property with respect to the singular perturbation.

(g)*State observer design*

In this thesis, there is an underlying hypothesis of full state measurement. In cases for which this hypothesis is invalid, identification of conditions under which state observers may be constructed is a further topic for study.

(h)*Optimal control systems*

Feedback controlled differential inclusion systems subject to appropriate performance indices could be investigated with the view of characterizing classes of systems which can be optimized in the presence of uncertainty.

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